

# ANALYTIC SKEW-PRODUCTS OF QUADRATIC POLYNOMIALS OVER MISIUREWICZ-THURSTON MAPS

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**ABSTRACT.** We consider skew-products of quadratic maps over certain Misiurewicz-Thurston maps and study their statistical properties. We prove that, when the coupling function is a polynomial of odd degree, such a system admits two positive Lyapunov exponents almost everywhere and a unique absolutely continuous invariant probability measure.

## 1. INTRODUCTION

A quadratic polynomial  $Q_c(x) = c - x^2$  ( $1 < c \leq 2$ ) induces a unimodal map on the interval  $[c - c^2, c]$ . If the critical point 0 is strictly pre-periodic under iteration of  $Q_c$ , then we say that  $Q_c$  is *Misiurewicz-Thurston*. These maps are the simplest non-uniformly expanding dynamical systems, and their properties are well understood. See for example [8]. When considered as a holomorphic map defined on the Riemann Sphere, the complex dynamics generated by  $Q_c$  is also exhaustively studied. In this article, we shall make use of its subhyperbolicity as was studied in, for example, [6, §V.4] or [7, §19].

In Viana [11], Misiurewicz-Thurston quadratic polynomials were used to construct non-uniformly expanding maps in dimension greater than one. He considered the following skew-product

$$G : (\mathbb{R}/\mathbb{Z}) \times \mathbb{R} \cup, \quad (\theta, y) \mapsto (d \cdot \theta, Q_c(y) + \alpha \sin(2\pi\theta)),$$

where  $d \geq 16$  is an integer, and  $Q_c$  ( $1 < c < 2$ ) is Misiurewicz-Thurston. He proved that  $G$  has two positive Lyapunov exponents almost everywhere, provided that  $\alpha > 0$  is small enough. The assumption that  $d \geq 16$  was weakened to  $d \geq 2$  in [5]. See also [9] for a similar result in non-integer case of  $d$ .

In Schnellmann [10], the following skew product was considered:

$$\mathcal{F} : [a - a^2, a] \times \mathbb{R} \cup, \quad (x, y) \mapsto (g(x), Q_b(y) + \alpha\varphi(x)), \quad (1)$$

where  $g = Q_a^{m_1}$ ,  $Q_a$  ( $1 < a \leq 2$ ) and  $Q_b$  ( $1 < b < 2$ ) are Misiurewicz-Thurston, and  $m_1$  is a large positive integer. For certain coupling function  $\varphi$ , with singularity and depending on  $a$ , the author proved that  $\mathcal{F}$  has two positive Lyapunov exponents almost everywhere, provided that  $\alpha > 0$  is small enough.

In this paper, we shall also consider systems in the form of (1), but the coupling function  $\varphi$  will be taken to be a nonconstant polynomial independent of  $a$ . The main result is the following:

**Main Theorem.** *Let  $Q_a$  ( $1 < a \leq 2$ ) and  $Q_b$  ( $1 < b < 2$ ) be Misiurewicz-Thurston and let  $\varphi$  be a polynomial of odd degree. Assume also that  $Q_a$  is topologically exact on  $[a - a^2, a]$ . Then there exists a positive integer  $m_0 = m_0(a)$  such that for each positive integer  $m_1 \geq m_0$  the following holds: For any  $\alpha > 0$  sufficiently small, the map  $\mathcal{F}$  defined in (1) has two positive Lyapunov exponents. Moreover,  $\mathcal{F}$  has a unique invariant probability measure that is absolutely continuous with respect to the Lebesgue measure.*

*Remark.* The assumption that  $\varphi$  is of odd degree seems quite artificial. Unfortunately, in our argument, this assumption cannot be fully gotten rid of. See the proof of the claim in Lemma 2.6 and the remark after that lemma for details.

We shall use many ideas appearing in the works cited above. In particular, we choose  $m_0$  large enough, so that the map  $\mathcal{F}$  is *partially hyperbolic*, i.e. the expansion along the  $x$ -direction is stronger than the  $y$ -direction. The exact choice of  $m_0$  will be specified in (5), at the end of §2.1.

To better visualize the partial hyperbolicity, we shall introduce a conjugation map  $u : [a - a^2, a] \rightarrow I_a$  in §2.2, so that the conjugated map  $h_0 = u \circ Q_a \circ u^{-1} : I_a \cup$  becomes uniformly expanding. Our construction of  $u$  is based on subhyperbolicity of  $Q_a$  near its Julia set. A similar construction, using instead the absolutely continuous invariant density, was used in [10]. We shall show that  $u^{-1}$  and inverse branches of  $h_0$  have very nice analytic properties, see Lemma 2.3. Then the map  $\mathcal{F}$  defined in (1) can be conjugate to the following map  $F$  via  $u \times \text{id}_{\mathbb{R}}$ :

$$F : I_a \times \mathbb{R} \cup \quad , \quad (\theta, y) \mapsto (h(\theta), Q_b(y) + \alpha\phi(\theta)), \quad (2)$$

where  $h = h_0^{m_1}$  and  $\phi = \varphi \circ u^{-1}$ . We shall prove the equivalent statement of the Main Theorem for  $F$  instead of  $\mathcal{F}$ . An important advantage of this construction is that in the  $\theta$ -coordinate, the derivative of an admissible curve, i.e., the image of a horizontal curve under iteration of  $F$ , can be approximated by an analytic function  $\alpha T$ , for some  $T$  chosen from a certain compact space of holomorphic maps.

The basic strategy to prove our Main Theorem for the system  $F$  is the same as that in previous works [11, 5, 10]. The main effort is to control recurrence of typical orbits to the critical line  $y = 0$ , see Proposition 4.3. This proposition, together with the “Building Expansion” Lemmas in [11], which are summarized in Lemma 2.7, implies that  $F$  is non-uniformly expanding in the sense of [3], and hence the results proved therein complete the proof of our Main Theorem.

Following Viana’s original argument, Proposition 4.3 is reduced to showing that under iteration of  $F$ , a horizontal curve becomes *non-flat* and widely spreads in the  $y$ -direction. In our argument, these two properties are obtained in Proposition 3.1 and Lemma 4.1 respectively. Lemma 2.6, serving as an intermediate step, is proved by combining subhyperbolicity of  $Q_a$  with the super attracting behavior of  $Q_a$  near infinity.

It remains an interesting open problem whether the statement of our Main Theorem holds for  $(a, b)$  chosen from a positive subset of  $[1, 2] \times [1, 2)$ .

## 2. PRELIMINARIES

**2.1. Subhyperbolicity of  $Q_a$ .** Let us review some useful properties of the complex dynamics of the Misiurewicz-Thurston quadratic polynomial  $Q_a (1 < a \leq 2)$ , which are well discussed in many literatures, see for example [6, §V.3] or [7, §19].

Denote the *post-critical* set of  $Q_a$  by  $PC_a$ , i.e.

$$PC_a = \{ Q_a^n(0) : n \geq 1 \}.$$

Denote the *Julia* set of  $Q_a$  by  $\mathcal{J}_a$ . Since  $Q_a$  is Misiurewicz-Thurston, its *Fatou* set  $\overline{\mathbb{C}} \setminus \mathcal{J}_a$  coincides with the attracting basin of infinity. Therefore, we can fix some  $R = R(a) > 0$ , and denote  $V_a = \{ z \in \mathbb{C} : |z| < R \}$ , such that

$$\mathcal{J}_a \subset V_a \quad \text{and} \quad Q_a^{-1}(\overline{V_a}) \subset V_a.$$

An important feature of  $Q_a$  as a Misiurewicz-Thurston map is the so-called *subhyperbolicity*. That is to say, there exists a conformal metric  $\rho_a(z)|dz|$  around  $\mathcal{J}_a$  (with mild singularities at  $PC_a$ ), such that with respect to this metric,  $Q_a$  is uniformly expanding in some neighborhood of  $\mathcal{J}_a$ . We shall give an explicit construction of  $\rho_a(z)|dz|$  below for further usage. For this purpose, let us start with the following lemma. Because this lemma can be essentially found in [6, §V, Theorem 3.1] or [7, §19], we merely provide an outline of its proof.

**Lemma 2.1.** *For each Misiurewicz-Thurston  $Q_a$  ( $1 < a \leq 2$ ), there exists a branched covering  $\pi : \mathbb{D} \rightarrow V_a$ , where  $\mathbb{D}$  denotes the unit disk in  $\mathbb{C}$ , satisfying the following properties:*

- (i) *The collection of critical points of  $\pi$  is precisely equal to  $\pi^{-1}(PC_a)$ , and for each  $\hat{c} \in \pi^{-1}(PC_a)$ ,  $\pi''(\hat{c}) \neq 0$ , i.e. the local degree of  $\pi$  is constantly equal to 2 on  $\pi^{-1}(PC_a)$ .*
- (ii)  *$\pi$  is normal. That is to say,  $\pi|_{\mathbb{D} \setminus \pi^{-1}(PC_a)}$  is normal as an unbranched covering, i.e. for each  $p \in V_a \setminus PC_a$ , the covering transformation group of  $\pi|_{\mathbb{D} \setminus \pi^{-1}(PC_a)}$  acts transitively on  $\pi^{-1}(p)$ .*
- (iii) *For each pair of points  $(\hat{p}, \hat{q})$  in  $\mathbb{D}$  with  $Q_a(\pi(\hat{p})) = \pi(\hat{q})$ ,  $Q_a^{-1}$  can be lifted to a single-valued holomorphic map  $\hat{\tau} : \mathbb{D} \cup \{\infty\} \rightarrow \mathbb{D} \cup \{\infty\}$ , which maps  $\hat{q}$  to  $\hat{p}$ . Moreover,  $\hat{c} \in \mathbb{D}$  is a critical point of  $\hat{\tau}$  precisely if  $\pi(\hat{c}) \in PC_a \setminus \{a\}$  and  $\pi(\hat{\tau}(\hat{c})) \notin PC_a$ . Besides, once  $\hat{\tau}'(\hat{c}) = 0$ , then  $\hat{\tau}''(\hat{c}) \neq 0$ .*

*Sketch of Proof.* Fix  $q \in V_a \setminus PC_a$  as a base point. For each  $v \in PC_a$ , let  $l_v \in \pi_1(V_a \setminus PC_a, q)$  be represented by a simple closed curve based on  $q$  whose winding number around  $v$  is one and whose winding number around  $v'$  is 0 for each  $v' \in PC_a \setminus \{v\}$ . Then  $\pi_1(V_a \setminus PC_a, q)$  is the free group generated by  $\{l_v\}_{v \in PC_a}$ . Let  $G$  be its normal subgroup generated by  $\{l_v^2\}_{v \in PC_a}$ . Then there exists a normal unbranched covering  $\pi : \mathbb{D}/G \rightarrow V_a \setminus PC_a$ . For any small disk  $B(v, r)$ ,  $v \in PC_a$ , each connected component of  $\pi^{-1}(B(v, r) \setminus \{v\})$  is a punctured disk, and the restriction of  $\pi$  to it is a double covering. By filling up the punctured points into  $\mathbb{D}/G$ , it is easy to verify that  $\pi$  can be extended to a branched covering from some simply connected Riemann surface to  $V_a$ . Since  $V_a$  is hyperbolic, we obtain a branched covering  $\pi : \mathbb{D} \rightarrow V_a$ . According to the construction of  $\pi$ , assertions (i) and (ii) hold automatically.

Because  $Q_a$  has a unique critical point 0 with  $Q_a''(0) \neq 0$ , the construction of  $\pi$  exactly guarantees that  $Q_a^{-1}$  can be locally lifted to a holomorphic map from  $\mathbb{D}$  to  $\mathbb{D}$  with respect to  $\pi$ . Since  $\mathbb{D}$  is simply connected, the global existence of the lift follows from monodromy theorem. It remains to check the statement about critical points of  $\hat{\tau}$ . Since  $Q_a \circ \pi \circ \hat{\tau} = \pi$  and since  $\pi'(\hat{c}) = 0$  implies that  $\pi''(\hat{c}) \neq 0$ ,  $\hat{c}$  is a critical point of  $\hat{\tau}$  if and only if  $\hat{c}$  is a critical point of  $\pi$  and  $\hat{\tau}(\hat{c})$  is not a critical point of  $Q_a \circ \pi$ . The conclusion follows.  $\square$

Let  $\rho_{\mathbb{D}}(w)|dw|$  be the standard Poincaré metric on  $\mathbb{D}$ , where  $\rho_{\mathbb{D}}(w) = \frac{2}{1-|w|^2}$ . By (i) and (ii) of Lemma 2.1,  $\rho_{\mathbb{D}}(w)|dw|$  can be pushed forward by  $\pi$  to  $V_a$ , which induces a metric  $\rho_a(z)|dz|$  on  $V_a$ , i.e.

$$\rho_{\mathbb{D}}(w) = \rho_a(\pi(w))|\pi'(w)|, \quad \forall w \in \mathbb{D} \setminus \pi^{-1}(PC_a). \quad (3)$$

Therefore,  $\rho_a$  is a strictly positive analytic function on  $V_a \setminus PC_a$ , and it has a singularity of order  $|z - v|^{-\frac{1}{2}}$  at  $v$  for each  $v \in PC_a$ . It follows that the function  $\frac{\rho_a(Q_a(z))}{\rho_a(z)}|(Q_a)'(z)|$  is actually continuous on  $Q_a^{-1}(V_a)$ . Moreover, by (iii) of Lemma 2.1 and Schwarz lemma, it is strictly larger than 1 on  $Q_a^{-1}(V_a)$ . In particular, since  $Q_a^{-1}(\overline{V_a})$  is a compact subset of  $V_a$ ,

$$\lambda_a := \inf_{z \in Q_a^{-1}(\overline{V_a})} \frac{\rho_a(Q_a(z))}{\rho_a(z)} |(Q_a)'(z)| > 1. \quad (4)$$

Here is the right position to specify our choice of  $m_0$  in the statement of the Main Theorem. By definition,  $m_0$  is the minimal positive integer such that

$$\lambda_a^{m_0} > 4. \quad (5)$$

By considering the derivative of  $Q_a$  at its fixed point  $\frac{\sqrt{1+4a}-1}{2}$ , it follows that

$$\lambda_a \leq \sqrt{1+4a} - 1 \leq 2.$$

For some technical reason, we shall also need a constant  $\tilde{\lambda}_a \in (4^{\frac{1}{m_0}}, \lambda_a)$ . To be definite, let

$$\tilde{\lambda}_a := \frac{\lambda_a + 4^{\frac{1}{m_0}}}{2}.$$

*Remark.* In fact our argument is valid for a smaller  $m_0$ . On the one hand, we can replace  $\lambda_a$  with a larger number  $\mu_a > 1$ , which only needs to satisfy that

$$\rho_a(Q_a^n(x))|(Q_a^n)'(x)| \geq C_a \mu_a^n \cdot \rho_a(x), \quad \forall x \in [a - a^2, a], \forall n \geq 1,$$

for some  $C_a > 0$ . On the other hand, the upper bound 4 of  $|Q'_b|$  can be replaced by some  $1 < R_b < 2$ , which only needs to satisfy that

$$|(Q_b^n)'(y)| \leq C_b R_b^n, \quad \forall y \in \widehat{I}_b, \forall n \geq 0,$$

for some  $C_b > 0$ . See [5, Lemma 3.1] for the existence of  $R_b$ . Then to guarantee that our argument works,  $m_0$  only needs to satisfy that

$$\mu_a^{m_0} > R_b,$$

but several details will be more redundant. For simplification, let us work with the assumption  $\lambda_a^{m_0} > 4$ .

**2.2. Expanding coordinates.** Define

$$u = u_a : [a - a^2, a] \rightarrow I_a, \quad x \mapsto \int_0^x \rho_a(t) dt,$$

where

$$I_a := u([a - a^2, a]).$$

Let  $h_0$  be the conjugated map of  $Q_a : [a - a^2, a] \hookrightarrow [a - a^2, a]$  via  $u$ , i.e.

$$h_0 : I_a \rightarrow I_a, \quad h_0 = u \circ Q_a \circ u^{-1}.$$

Note that  $u(0) = 0$  is the unique turning point of  $h_0$ . By definition,

$$|h_0'(u(x))| = \frac{\rho_a(Q_a x)}{\rho_a(x)} |Q_a'(x)| \geq \lambda_a, \quad \forall x \in [a - a^2, a]. \quad (6)$$

For every  $n \geq 0$ , let

$$Q_n := \{\omega \subset I_a : \omega \text{ is a connected component of } I_a \setminus h_0^{-n}(u(PC_a))\}.$$

Then  $\{Q_n\}_{n \geq 0}$  is a sequence of nested Markov partitions of  $h_0$ . To piecewise analytically continue  $h_0^{-1}$  on each  $\omega \in Q_0$ , let us consider the following composition of maps

$$\hat{u} : \pi^{-1}([a - a^2, a]) \rightarrow I_a, \quad \hat{p} \mapsto u(\pi(\hat{p})).$$

Given a pair  $(\omega_0, \omega_1) \in Q_0 \times Q_1$  with  $h_0(\omega_1) = \omega_0$ , denote  $\tau = (h_0|_{\omega_1})^{-1}$ ,  $J_i = u^{-1}(\omega_i)$ ,  $i = 0, 1$ , and let  $\sigma : J_0 \rightarrow J_1$  be the corresponding branch of  $Q_a^{-1}$ . For  $i = 0, 1$ , let  $\hat{\omega}_i$  be an arbitrary connected component of  $\pi^{-1}(J_i)$ , and let  $\hat{\tau} : \hat{\omega}_0 \rightarrow \hat{\omega}_1$  be the lift of  $\sigma$  via  $\pi$ . Then we have the following commutative diagram.

$$\begin{array}{ccc}
\hat{\omega}_0 & \xrightarrow{\hat{\tau}} & \hat{\omega}_1 \\
\pi \downarrow & & \downarrow \pi \\
J_0 & \xrightarrow{\sigma} & J_1 \\
u \downarrow & & \downarrow u \\
\omega_0 & \xrightarrow{\tau} & \omega_1
\end{array} \tag{7}$$

**Lemma 2.2.** *For the notations introduced above, the following statements hold.*

- (i)  $\hat{u}$  can be analytically continued to a univalent holomorphic function on some open neighborhood of  $\hat{\omega}_0$  in  $\mathbb{D}$ .
- (ii)  $\tau$  can be analytically continued to a holomorphic function on some open neighborhood of  $\overline{\omega_0}$  in  $\mathbb{C}$ . Moreover,  $c \in \overline{\omega_0}$  is a critical point of  $\tau$  precisely if  $c \in \partial\omega_0$ ,  $c \neq u(a)$  and  $\tau(c) \notin u(PC_a)$ . Besides, once  $\tau'(c) = 0$ , then  $\tau''(c) \neq 0$ .

*Proof.* Since  $u' = \rho_a$ , by (3),  $u' > 0$  on  $J_0$ , and  $u$  is real analytic on  $J_0$  with singular points of order  $\frac{1}{2}$  at both end points. Due to (i) of Lemma 2.1,  $\pi' \neq 0$  on  $\hat{\omega}_0$  and  $\pi$  has critical points of local degree 2 at both end points of  $\hat{\omega}_0$ . Therefore, it is easy to see that assertion (i) holds.

Since  $\tau = \hat{u} \circ \hat{\tau} \circ \hat{u}^{-1}$  on  $\omega_0$ , assertion (ii) follows immediately from (iii) of Lemma 2.1 and assertion (i). □

Given a bounded interval  $I \subset \mathbb{R}$  and  $\xi > 0$ , let

$$B_\xi(I) = \{z \in \mathbb{C} : \text{dist}(z, I) < \xi\}.$$

The following two lemmas are the main results of this subsection, which motivate us to replace  $g$  with  $h = u \circ g \circ u^{-1}$  as our base dynamics for taking advantage of nice analytical properties of  $h^{-1}$ .

**Lemma 2.3.** *There exist  $\xi > 0$  and  $D_{\xi,i} > 0$ ,  $\forall i \geq 1$ , such that the following statements hold.*

- (i) Given  $\omega \in \mathcal{Q}_0$ ,  $u^{-1}|_\omega$  can be analytically continued to a holomorphic function on  $B_\xi(\omega)$ .
- (ii) Given  $(\omega_0, \omega_1) \in \mathcal{Q}_0 \times \mathcal{Q}_1$  with  $h_0(\omega_1) = \omega_0$ ,  $(h_0|_{\omega_1})^{-1}$  can be analytically continued to a holomorphic function  $\tau$  on  $B_\xi(\omega_0)$  and  $|\tau'| \leq \tilde{\lambda}_a^{-1}$  on  $B_\xi(\omega_0)$ . Moreover, for each  $0 < \xi' \leq \xi$  and each interval  $I \subset \omega_0$ ,  $\tau(B_{\xi'}(I)) \subset B_{\xi'}(\tau(I))$ .
- (iii) Given  $n \geq 1$  and  $(\omega_0, \omega_n) \in \mathcal{Q}_0 \times \mathcal{Q}_n$  with  $h_0^n(\omega_n) = \omega_0$ ,  $(h_0^n|_{\omega_n})^{-1}$  can be analytically continued to a holomorphic function  $\tau_n : B_\xi(\omega_0) \rightarrow B_\xi(\omega_n)$ . Moreover,

$$\sup_{z \in B_\xi(\omega_0)} |\tau_n^{(i)}(z)| \leq D_{\xi,i} \tilde{\lambda}_a^{-n}, \quad \forall i \geq 1. \tag{8}$$

*Proof.* Let  $\hat{\omega}$  be an arbitrary connected component of  $\hat{u}^{-1}(\omega)$ . Then  $u^{-1} = \pi \circ (\hat{u}|_{\hat{\omega}})^{-1}$  on  $\omega$ , and hence (i) of Lemma 2.2 implies that  $u^{-1}$  can be analytically continued to a holomorphic function on some open neighborhood of  $\overline{\omega}$  in  $\mathbb{C}$ . Since  $\mathcal{Q}_0$  is a finite set, assertion (i) holds for some  $\xi = \xi_1 > 0$ .

On the one hand, according to (ii) of Lemma 2.2,  $(h_0|_{\omega_1})^{-1}$  can be extended to a holomorphic function  $\tau$  on some open neighborhood of  $\overline{\omega_0}$ . On the other hand, (6) implies that

$|\tau'| \leq \lambda_a^{-1} < \tilde{\lambda}_a^{-1}$  on  $\omega_0$ . Noting that  $\mathcal{Q}_0 \times \mathcal{Q}_1$  is a finite set, there exists  $\xi_2 > 0$ , independent of  $(\omega_0, \omega_1)$ , such that  $\tau$  is holomorphic on  $B_{\xi_2}(\omega_0)$  and  $|\tau'| \leq \tilde{\lambda}_a^{-1}$  on  $B_{\xi_2}(\omega_0)$ . For each  $0 < \xi' \leq \xi_2$  and each interval  $I \subset \omega_0$ ,  $B_{\xi'}(I)$  is a convex subset of  $B_{\xi_2}(\omega_0)$ . It follows that  $|\tau'| \leq \tilde{\lambda}_a^{-1} < 1$  on  $B_{\xi'}(I)$ , and consequently  $\tau(B_{\xi'}(I)) \subset B_{\xi'}(\tau(I))$ . This completes the proof of (ii) for  $\xi = \xi_2$ .

Consider  $(h_0^n|_{\omega_n})^{-1}$  as a composition of  $n$  copies of  $h_0^{-1}$  and apply (ii) repeatedly for  $\xi = \xi_2$ . Therefore, it can be extended to a holomorphic function  $\tau_n : B_{\xi_2}(\omega_0) \rightarrow B_{\xi_2}(\omega_n)$  and  $|\tau_n| \leq \tilde{\lambda}_a^{-n}$  on  $B_{\xi_2}(\omega_0)$ . Take  $\xi = \min(\xi_1, \frac{\xi_2}{2})$ , so that both (i) and (ii) hold simultaneously. (8) follows from Cauchy's estimate.  $\square$

**Lemma 2.4.** *There exists  $C_d > 0$ , such that for any  $\omega_n \in \mathcal{Q}_n (n \geq 1)$  and any measurable set  $E \subset \omega_n$ , we have*

$$C_d^{-1} \cdot |h_0^n(E)|^2 \leq \frac{|E|}{|\omega_n|} \leq C_d \cdot |h_0^n(E)|. \quad (9)$$

*Proof.* Let  $N = N(a) \geq 1$  be the minimal integer such that for each  $\omega \in \mathcal{Q}_N$ ,  $\partial\omega$  contains at most one point in  $h_0^{-1}(u(PC_a))$ . For each  $n \geq 0$ , denote  $\mathcal{Q}_{n+N}$  by  $\tilde{\mathcal{Q}}_n$ . It suffices to prove the lemma for  $\{\tilde{\mathcal{Q}}_n\}_{n \geq 0}$  instead of  $\{\mathcal{Q}_n\}_{n \geq 0}$ , because of the reasons below:

- there exists an integer  $M = M(a)$ , such that for each  $n \geq 0$ , every element in  $\mathcal{Q}_n$  is a union of no more than  $M$  elements in  $\tilde{\mathcal{Q}}_n$  with a finite set;
- once the left inequality in (9) has been proved for  $\{\tilde{\mathcal{Q}}_n\}_{n \geq 0}$ , it follows that there exists  $p > 0$ , such that for every  $n \geq 0$  and every pair  $(\omega, \tilde{\omega}) \in \mathcal{Q}_n \times \tilde{\mathcal{Q}}_n$  with  $\tilde{\omega} \subset \omega$ , we have  $|\tilde{\omega}| \geq p|\omega|$ .

To begin with, recall the statement in (ii) of Lemma 2.3, which says that, for each pair  $(J_0, J_1) \in \tilde{\mathcal{Q}}_0 \times \tilde{\mathcal{Q}}_1$  with  $h_0(J_1) = J_0$ ,  $(h_0|_{J_1})^{-1}$  can be extended to a holomorphic function  $\zeta : B_{\xi}(J_0) \rightarrow B_{\xi}(J_1)$ . Moreover, for each  $0 < \xi' \leq \xi$ , and each subinterval  $I$  of  $J_0$ ,  $\zeta(B_{\xi'}(I)) \subset B_{\xi'}(\zeta(I))$ . By the choice of  $\tilde{\mathcal{Q}}_0$ ,  $\partial J_0 \cap u(PC_a)$  contains at most one point. According to (ii) of Lemma 2.2, we can fix  $0 < \xi' \leq \xi$  small, independent of  $(J_0, J_1)$ , such that:

- if  $\partial J_0 \cap u(PC_a) = \{c\}$ ,  $c \neq u(a)$  and  $\zeta(c) \notin PC_a$ , then  $c$  is the unique critical point of  $\zeta$  in  $B_{\xi'}(J_0)$  and  $\zeta''(c) \neq 0$ ;
- otherwise,  $\zeta : B_{\xi'}(J_0) \rightarrow B_{\xi'}(J_1)$  is univalent.

Given  $\omega_n \in \tilde{\mathcal{Q}}_n$ , for each  $0 \leq i \leq n-1$ , let  $\omega_i = h_0^{n-i}(\omega_n)$ , let  $J_i$  be the unique element in  $\tilde{\mathcal{Q}}_0$  containing  $\omega_i$ , and let  $\zeta_i : B_{\xi'}(J_{i-1}) \rightarrow B_{\xi'}(J_i)$  be the analytic continuation of  $(h_0|_{\omega_i})^{-1}$ . Then  $(h_0^n|_{\omega_n})^{-1}$  can be extended to a holomorphic function  $\tau_n : B_{\xi'}(\omega_0) \rightarrow B_{\xi'}(\omega_n)$ , where  $\tau_n = \zeta_n \circ \dots \circ \zeta_1$ . To proceed, a classified discussion of the possibility of  $\{\partial\omega_i \cap u(PC_a)\}_{i=0}^n$  is needed. There are four cases.

1.  $\partial\omega_0 \cap u(PC_a) = \emptyset$ . Denote  $\sigma_n = u^{-1} \circ \tau_n \circ u = (Q_a^n|_{u^{-1}(\omega_n)})^{-1}$ . Let  $J$  be the element in  $\mathcal{Q}_0$  containing  $\omega_0$ . By definition,  $\sigma_n$  can be extended to a univalent holomorphic function on  $\mathbb{C}_{u^{-1}(J)} := (\mathbb{C} \setminus \mathbb{R}) \cup u^{-1}(J)$ . For each  $\delta > 0$ , denote  $B_{\delta}(\omega_0) \cap \mathbb{R}$  by  $I_{\delta}$ . Since  $(u^{-1})' \neq 0$  on  $J$  and since  $\tilde{\mathcal{Q}}_0$  is a finite set, there exists  $\delta_0 > 0$ , independent of  $\omega_0$ , such that  $u^{-1}$  can be extended to a univalent holomorphic function on  $B_{2\delta_0}(\omega_0)$ , which satisfies that  $I_{2\delta_0} \subset J$  and  $u^{-1}(B_{2\delta_0}(\omega_0)) \subset \mathbb{C}_{u^{-1}(J)}$ . Then  $\sigma_n \circ u^{-1}$  is univalent on  $B_{2\delta_0}(\omega_0)$ , so by Koebe distortion theorem, the distortion of  $\sigma_n \circ u^{-1}$  on  $B_{\delta_0}(\omega_0)$  is only dependent on  $\delta_0$ . It follows that there exists  $C > 0$ , determined by  $\delta_0$  only, such that for either component of  $\sigma_n \circ u^{-1}(I_{\delta_0} \setminus \omega_0)$ , its length is no less than  $C \cdot |u^{-1}(\omega_n)|$ , where  $u^{-1}(\omega_n) = \sigma_n \circ u^{-1}(\omega_0)$ . Also note that  $\sigma_n \circ u^{-1}(I_{\delta_0})$  does not intersect  $PC_a$ , the singular set of  $u$ . Because all the singularity points of  $u$  are of

square root type, we can conclude that there exists  $\epsilon > 0$ , only dependent on  $\delta_0$ , such that  $u$  can be extended to a univalent function on  $B_\epsilon(u^{-1}(\omega_n))$ . Since  $\sigma_n \circ u^{-1}$  has bounded distortion on  $B_{\delta_0}(\omega_0)$ , finally we can find  $0 < \delta \leq \delta_0$ , determined by  $\epsilon$  only, such that  $\tau_n = u \circ \sigma_n \circ u^{-1}$  is univalent on  $B_\delta(\omega_0)$ . Then the conclusion follows from Koebe distortion theorem.

2.  $\partial\omega_n \cap u(PC_a) \neq \emptyset$ . Then by the choice of  $\xi'$ ,  $\zeta_i : B_{\xi'}(J_{i-1}) \rightarrow B_{\xi'}(J_i)$  is univalent,  $\forall 1 \leq i \leq n$ . Therefore,  $\tau_n : B_{\xi'}(\omega_0) \rightarrow B_{\xi'}(\omega_n)$  is univalent, and hence the conclusion also follows from Koebe distortion theorem.
3. There exists  $1 \leq i \leq n$ , such that  $0 \in \partial\omega_i$ . Then  $\partial\omega_j \cap u(PC_a)$  consists of a unique point  $Q_a^{i-j}(0)$ ,  $j = 0, 1, \dots, i$ , and according to the definition of  $\tilde{Q}_0$ ,  $J_i \cap u(PC_a) = \emptyset$ . Therefore,  $\zeta_i \circ \dots \circ \zeta_1 : B_{\xi'}(\omega_0) \rightarrow B_{\xi'}(\omega_i)$  is in the situation of Case 2, i.e. it is univalent;  $\zeta_n \circ \dots \circ \zeta_{i+1} : B_{\xi'}(J_i) \rightarrow B_{\xi'}(J_n)$  is in the situation of Case 1, so it is univalent on  $B_\delta(J_i)$ . As a result, for  $\delta' = \min(\xi', \delta)$ ,  $\tau_n$  is univalent on  $B_{\delta'}(\omega_0)$ . Once again, the conclusion is implied by Koebe distortion theorem.
4. There exists  $1 \leq i \leq n$ , such that  $\partial\omega_{i-1} \cap u(PC_a) = \{c\}$ ,  $c \neq u(a)$  and  $\partial\omega_i \cap u(PC_a) = \emptyset$ . Note that  $\zeta_i(c) \in \partial\omega_i \cap h_0^{-1}(u(PC_a)) \subset \partial J_i$ , by the definition of  $\tilde{Q}_0$ , which implies that  $\partial J_i \cap u(PC_a) = \emptyset$ . On the one hand, by the choice of  $\xi'$ ,  $\zeta_i : B_{\xi'}(J_{i-1}) \rightarrow B_{\xi'}(J_i)$  has a unique critical point  $c$  with  $\zeta_i''(c) \neq 0$ . It implies that there is a constant  $C > 1$ , only dependent on  $\xi'$ , such that for every measurable set  $E_i \subset \omega_i$ , we have

$$C^{-1} \frac{|h_0(E_i)|^2}{|\omega_{i-1}|^2} \leq \frac{|E_i|}{|\omega_i|} \leq C \frac{|h_0(E_i)|}{|\omega_{i-1}|}.$$

On the other hand, similar to the discussion in Case 3,  $\zeta_{i-1} \circ \dots \circ \zeta_1 : B_{\xi'}(\omega_0) \rightarrow B_{\xi'}(\omega_{i-1})$  is in the situation of Case 2, and  $\zeta_n \circ \dots \circ \zeta_{i+1} : B_{\xi'}(J_i) \rightarrow B_{\xi'}(J_n)$  is in the situation of Case 1. Then by Koebe distortion theorem, there exists a constant  $\tilde{C} > 1$ , only dependent on  $\delta'$ , such that for any measurable sets  $E_{i-1} \subset \omega_{i-1}$  and  $E_n \subset \omega_n$ ,

$$\tilde{C}^{-1} |h_0^{i-1}(E_{i-1})| \leq \frac{|E_{i-1}|}{|\omega_{i-1}|} \leq \tilde{C} |h_0^{i-1}(E_{i-1})|, \quad \tilde{C}^{-1} \frac{|h_0^{n-i}(E_n)|}{|\omega_i|} \leq \frac{|E_n|}{|\omega_n|} \leq \tilde{C} \frac{|h_0^{n-i}(E_n)|}{|\omega_i|}.$$

Combining the estimates above, (9) follows in this case.  $\square$

**2.3. Backward shrinking of  $Q_a$ .** For further usage, let us give some detailed estimates on the accumulation rate of points in  $\mathbb{C} \setminus \mathcal{J}_a$  to  $\mathcal{J}_a$  under iterated action of  $Q_a^{-1}$ .

**Lemma 2.5.** *For each  $m \in \mathbb{N}$ , there exists  $K_m > 0$ , such that the following statement holds. Let  $z_0 \in \mathbb{C} \setminus \mathcal{J}_a$  and  $z_i := Q_a^{mi}(z_0)$ ,  $\forall i \geq 0$ . Assume that  $z_n \notin V_a$  for some  $n \in \mathbb{N}$  and let  $\sigma$  be the branch of  $Q_a^{-mn}$  with  $\sigma(z_n) = z_0$ . Then there are two cases:*

(i) *If  $z_1 \notin Q_a^{-1}(V_a)$ , then*

$$|z_0| \leq K_m |z_n|^{2^{-mn}} \quad \text{and} \quad |\sigma'(z_n)| \leq K_m 2^{-mn} |z_n|^{2^{-mn}-1}. \quad (10)$$

(ii) *Otherwise, there exists  $0 < i < n$ , such that  $z_j \in Q_a^{-1}(V_a)$  for  $0 \leq j \leq i$  and  $z_j \notin Q_a^{-1}(V_a)$  for  $i < j \leq n$ , then*

$$|\sigma'(z_n)| \leq K_m 2^{-m(n-i)} \lambda_a^{-mi} |z_n|^{2^{-m(n-i)}-1}. \quad (11)$$

*Proof.* Denote  $F_m := \overline{\mathbb{C}} \setminus Q_a^{-m-1}(V_a)$ . Then  $F_m$  is a forward invariant compact subset of the Fatou set of  $Q_a$ . Note that on the Fatou set of  $Q_a$ ,  $\sigma$  can be conjugated to some branch of

$z \mapsto z^{2^{-m}}$  via the Böttcher coordinate about  $\infty$ . In case (i),  $z_i \in F_m, \forall 0 \leq i \leq n$ , and hence the conclusion follows from the compactness of  $F_m$ .

For case (ii), decompose  $\sigma$  into  $\sigma_1 \circ \sigma_2$ , where  $\sigma_1$  is the branch of  $Q_a^{-mi}$  with  $\sigma_1(z_i) = z_0$  and  $\sigma_2$  is the branch of  $Q_a^{-m(n-i)}$  with  $\sigma_2(z_n) = z_i$ . Note that  $z_j \in F_m$  when  $j \geq i$ . Therefore, as in case (i),  $|\sigma'_2(z_n)|$  has an upper bound of order  $2^{-m(n-i)} |z_n|^{2^{-m(n-i)}-1}$ . To estimate  $|\sigma'_1(z_i)|$ , because  $z_i = Q_a^{mi}(z_0) \in Q_a^{-1}(V_a)$ , we can apply (4) repeatedly to the orbit  $\{Q_a^j(z_0)\}_{j=0}^{mi-1}$ . As a result,

$$\rho_a(z_0)|\sigma'_1(z_i)| \leq \lambda_a^{-mi} \rho_a(z_i).$$

Since  $z_i \in F_m$ ,  $\rho_a(z_i)$  is bounded from above, so the conclusion in case (ii) follows.  $\square$

**2.4. A family of functions.** From now on, let us fix an arbitrary integer  $m_1 \geq m_0$  and choose  $g = Q_a^{m_1}$  as our base dynamics. As mentioned in the introduction, we shall study  $F$  defined in (2) rather than  $\mathcal{F}$ . Let us summarize some frequently used notations and their basic properties here.

- $\lambda_g := \lambda_a^{m_1} \in (4, 2^{m_1}]$  and  $h := h_0^{m_1} = u \circ g \circ u^{-1}$ . By definition,  $|h'| \geq \lambda_g$  on  $I_a$ .
- $\phi := \varphi \circ u^{-1}$  on  $I_a$ . Without loss of generality, assume that  $|\phi| \leq 1$  on  $I_a$ .
- $\widehat{I}_b := [-\sqrt{2b}, \sqrt{2b}]$ .  $Q_b$  maps  $I_b$  to its interior.
- Assume that  $\alpha$  is sufficiently small, so that  $F$  maps  $I_a \times \widehat{I}_b$  into itself.
- $\mathcal{P}_n := Q_{m_1 n}$ ,  $\forall n \geq 0$ . i.e.  $\{\mathcal{P}_n\}_{n \geq 0}$  is a nested sequence of Markov partitions of  $h$ .
- Denote  $F^n(\theta, y)$  by  $(h^n(\theta), f_n(\theta, y))$ . Due to (i) of Lemma 2.3 and the definition of  $F$ , for every  $\omega \in \mathcal{P}_0$  and every  $n \geq 1$ ,  $f_n$  is real analytic on  $\overline{\omega} \times \mathbb{R}$ .

Given a non-constant polynomial  $\varphi$ , let us introduce a family of functions as below, which is inspired by considering the  $\alpha$ -linear approximation of the derivative of high  $F$ -iteration of a horizontal curve. The importance of this family will become clear later. See Lemma 3.2 and Lemma 4.1.

**Definition 2.1.** Given  $\omega \in \mathcal{P}_0$ , an analytic function  $T : \omega \rightarrow \mathbb{R}$  is said to be in the family  $\mathcal{T}_\omega$ , if there exist a sequence  $\{\omega_n \in \mathcal{P}_n\}$  with  $\omega_0 = \omega$  and  $h(\omega_{n+1}) = \omega_n, \forall n \geq 0$ , and a sequence  $\{c_n \in \mathbb{R}\}$  with  $|c_n| \leq 4^{n-1}, \forall n \geq 2$ , such that

$$T(\theta) = (\phi \circ \tau_1)'(\theta) + \sum_{n=2}^{\infty} c_n (\phi \circ \tau_n)'(\theta), \quad \forall \theta \in \omega, \quad (12)$$

where  $\tau_n = (h^n|_{\omega_n})^{-1}$ .

Since in (12),  $|c_n| \leq 4^{n-1}$  and  $|\tau'_n(\theta)| \leq \lambda_g^{-n}$ , the series always converges. Due to (8), actually each  $T \in \mathcal{T}_\omega$  extends to a holomorphic function defined on  $B_\xi(\omega)$ .

All the useful properties of  $\mathcal{T}$  are listed in the lemma below. The following notations are used in its statement.

- $\omega_c$  denotes the unique element in  $\mathcal{P}_0$  containing 0.
- $\omega_c^\pm \subset \omega_c$  denote the only two elements in  $\mathcal{P}_1$  with  $0 \in \partial\omega_c^\pm$ .
- $\tau_c^\pm := (h|_{\omega_c^\pm})^{-1}$  are both defined on  $h(\omega_c^\pm) = h(\omega_c^-) \in \mathcal{P}_0$ .

By definition,  $\tau_c^\pm$  satisfy that

$$\gamma^\pm := u^{-1} \circ \tau_c^\pm \circ h \circ u : u^{-1}(\omega_c^\mp) \rightarrow u^{-1}(\omega_c^\pm), \quad x \mapsto -x. \quad (13)$$

**Lemma 2.6.** Let  $\varphi$  be a non-constant polynomial. For each  $\omega \in \mathcal{P}_0$ , the family  $\mathcal{T}_\omega$  defined for  $\varphi$ , considered as a space of holomorphic functions defined on  $B_\xi(\omega)$ , is compact with respect to the compact-open topology. Moreover,



(i) There exist  $A_n > 0$ ,  $n = 0, 1, 2, \dots$ , such that for each  $T \in \mathcal{T}_\omega$ ,

$$|T^{(n)}(\theta)| \leq A_n, \quad \forall \theta \in \omega, \forall n \geq 0. \quad (14)$$

(ii)  $0 \notin \mathcal{T}_\omega$ . More specifically, there exist  $l_0 \in \mathbb{N}$  and  $B > 0$ , such that for each  $T \in \mathcal{T}_\omega$ ,

$$\sum_{i=0}^{l_0-1} |T^{(i)}(\theta)| \geq 2B, \quad \forall \theta \in \omega. \quad (15)$$

(iii) If, additionally,  $\varphi$  is of odd degree, then for each  $T \in \mathcal{T}_{\omega_c}$  and each  $D \in [-4, 4]$ , the two functions

$$T^\pm := (\phi' + D \cdot T) \circ \tau_c^\pm \cdot (\tau_c^\pm)' \in \mathcal{T}_{h(\omega_c^\pm)} \quad (16)$$

are not identical to each other.

*Proof.* By (8) and (12), functions in the family  $\mathcal{T}_\omega$  are uniformly bounded on  $B_\xi(\omega)$ . Then according to Montel's theorem,  $\mathcal{T}_\omega$  is a pre-compact subset of holomorphic functions on  $B_\xi(\omega)$  with respect to the compact-open topology. On the other hand, in (12), for each  $n \geq 1$ , there are only finitely many choices of  $\tau_n$ , so by the definition of  $\mathcal{T}_\omega$ , it is also a closed subset. Therefore,  $\mathcal{T}_\omega$  is compact. Assertion (i) follows from the compactness of  $\mathcal{T}_\omega$  immediately.

To prove (ii) and (iii), let us change back to the  $x$  coordinate from the  $\theta$  coordinate to make use of the complex dynamics of  $Q_a$  on the whole Riemann sphere. (ii) and (iii) will be deduced from:

**Claim.** Given  $J \in u^{-1}(\mathcal{P}_0)$ , let  $\{\sigma_n\}_{n \geq 0}$  be a sequence of real analytic maps defined on  $J$  with  $\sigma_0 = \text{id}_J$  and  $g \circ \sigma_n = \sigma_{n-1}$ ,  $\forall n \geq 1$ , and let  $\{c_n\}_{n \geq 1}$  be a sequence of numbers with  $|c_n| \leq 4^n$ ,  $\forall n \geq 1$ . Then

$$S := \varphi' + \sum_{n=1}^{\infty} c_n (\varphi \circ \sigma_n)' \quad (17)$$

is not identically zero. Moreover, if  $\varphi$  is of odd degree and  $J = u^{-1}(\omega_c) \ni 0$ , then  $S$  is not an odd function on  $J \cap (-J)$ .

*Proof of Claim.* Note that  $S$  can be analytically continued to a holomorphic function defined on  $\mathbb{C}_J := (\mathbb{C} \setminus \mathbb{R}) \cup J$ . The basic idea is to show that around  $\infty$ , the  $\varphi'$  term in the series expression of  $S$  is dominating. To this end, take an arbitrary  $z \in \mathbb{C}_J \setminus V_a$ , and let  $n_z$  be the minimal integer such that  $\sigma_{n_z}(z) \in Q_a^{-1}(V_a)$ . The existence of  $n_z$  is guaranteed by the fact that the Fatou set of  $Q_a$  equals to the attracting basin of infinity, and the backward  $Q_a$ -invariance of  $V_a$  implies that  $\sigma_n(z) \notin Q_a^{-1}(V_a)$  if and only if  $n < n_z$ . Then we can apply Lemma 2.5 to  $z$  for  $m = m_1$ . Firstly, when  $n \leq n_z$ , by (10),

$$|(\varphi \circ \sigma_n)'(z)| \leq C \cdot 2^{-m_1 n} |z|^{\deg \varphi \cdot 2^{-m_1 n} - 1}.$$

Secondly, when  $n > n_z$ , by (11),

$$|(\varphi \circ \sigma_n)'(z)| \leq C \cdot 2^{-m_1 n_z} \lambda_g^{n_z - n} |z|^{2^{-m_1 n_z} - 1}.$$

Here  $C > 0$  is independent of  $z$  or  $n$ . Since  $|c_n| \leq 4^{n-1}$  and  $2^{m_1} \geq \lambda_g > 4$ , combining the two inequalities above, we have:

$$\left| \sum_{n=1}^{\infty} c_n (\varphi \circ \sigma_n)'(z) \right| \leq C' \cdot |z|^{\deg \varphi \cdot 2^{-m_1} - 1},$$

where  $C' > 0$  is independent of  $z$ . On the other hand, as  $|z| \rightarrow \infty$ ,  $|\varphi'(z)| \asymp |z|^{\deg \varphi - 1}$ . Therefore,

$$\lim_{\substack{z \in \mathbb{C}_J \\ |z| \rightarrow \infty}} \frac{S(z)}{\varphi'(z)} = 1. \quad (18)$$

Now we can complete the proof. Since  $\varphi$  is non-constant, (18) implies that  $S$  cannot be identically zero, i.e. the first statement of the claim holds. When  $\varphi$  is of odd degree,  $|\varphi'(z) + \varphi'(-z)|$  is bounded away from zero as  $|z| \rightarrow \infty$ . When  $0 \in J$ ,  $\mathbb{C}_{J \cap (-J)}$  is a domain containing 0 and symmetric about 0. These facts together with (18) imply that for  $z \in \mathbb{C}_{J \cap (-J)}$ ,  $|S(z) + S(-z)|$  is also bounded away from zero as  $|z| \rightarrow \infty$ . The second statement of the claim follows.  $\square$

Now let us prove (ii) and (iii) with the help of the claim. To start with, let us clarify the relation between the family  $\mathcal{T}$  and functions in the form of (17). Given  $\omega \in \mathcal{P}_0$  and  $T = (\phi \circ \tau_1)' + \sum_{n=2}^{\infty} c_n (\phi \circ \tau_n)' \in \mathcal{T}_\omega$ , denote  $J_1 = u^{-1}(\tau_1(\omega))$  and let  $S = T \circ (h \circ u) \cdot (h \circ u)'$  on  $J_1$ . By definition,  $\phi \circ \tau_n \circ h \circ u = \varphi \circ \sigma_{n-1}$ , where  $\sigma_0 = \text{id}_{J_1}$  and  $g \circ \sigma_n = \sigma_{n-1}$  on  $J_1$ ,  $\forall n \geq 1$ . It implies that  $S = \varphi' + \sum_{n=1}^{\infty} c_{n+1} (\varphi \circ \sigma_n)'$  has the form of (17) and it is automatically well defined on  $J \supset J_1$  with  $J \in u^{-1}(\mathcal{P}_0)$ .

To prove assertion (ii), for each  $T \in \mathcal{T}_\omega$ , let  $S = T \circ (h \circ u) \cdot (h \circ u)'$  on  $u^{-1}(\tau_1(\omega))$ . The claim says that  $S$  is not identically 0, which implies that  $0 \notin \mathcal{T}_\omega$ . The rest of assertion (ii) follows from this fact and the compactness of  $\mathcal{T}_\omega$  easily by reduction to absurdity.

To prove assertion (iii), by reduction to absurdity, for  $T^\pm$  appearing in (16), suppose that  $T^+ \equiv T^-$  on  $h(\omega_c^\pm)$ . Define  $S := (\phi' + D \cdot T) \circ u \cdot u'$  on  $J = u^{-1}(\omega_c)$ . By definition,  $S$  has the form of (17), and  $S = T^+ \circ (h \circ u) \cdot (h \circ u)'$  on  $u^{-1}(\omega_c^+)$ . By the assumption  $T^+ \equiv T^-$  on  $h(\omega_c^\pm)$ , we have:

$$S = T^- \circ (h \circ u) \cdot (h \circ u)' = S \circ \gamma^- \cdot (\gamma^-)' \quad \text{on } u^{-1}(\omega_c^+),$$

where  $\gamma^-$  is defined in (13) and  $\gamma^- = -\text{id}$  on  $u^{-1}(\omega_c^+)$ . It implies that  $S$  is an odd function on  $J \cap (-J)$ , which contradicts to the claim and completes the proof.  $\square$

*Remark.* It should be noted that without assuming that  $\varphi$  is of odd degree, the second statement of the claim in the proof of Lemma 2.6, and therefore assertion (iii) in Lemma 2.6, could fail in general. For example, given  $c \in [-4, 4]$ , let  $\varphi(x) = g(x) + cx$  and let  $S = \varphi' + \sum_{n=1}^{\infty} (-c)^n (\varphi \circ \sigma_n)'$  in the form of (17). Then actually  $S = g'$  is an odd function.

**2.5. Building expansion.** Here let us summarize some useful results from [11, Lemma 2.4, 2.5] and their proofs. It should be noted that these results are only based on the skew-product form of  $F$  and the Misiurewicz-Thurston property of  $Q_b$ , i.e. they are irrelevant to choice of  $h$  and  $\phi$ .

**Lemma 2.7.** *There exist constants  $\delta_* > 0$ ,  $C_* > 0$ ,  $1 < \sigma < 2$  and an integer  $N_\alpha$  with  $\sigma^{N_\alpha} \leq \alpha^{-1} \leq 4^{N_\alpha}$ , such that when  $\alpha$  is small, for an orbit  $(\theta_i, y_i) = F^i(\theta, y)$ ,  $(\theta, y) \in I_a \times \widehat{I}_b$ ,  $i = 0, 1, \dots$ , the following statements hold.*

- (i) *If  $|y| < 2\sqrt{\alpha}$ , then  $|y_k| \geq \delta_*$ ,  $k = 1, \dots, N_\alpha$ . Moreover, for each  $0 < \eta \leq \frac{1}{3}$ , when  $\alpha$  is small enough,  $\left| \frac{\partial f_{N_\alpha}}{\partial y}(\theta, y) \right| \geq |y| \alpha^{-1+\eta}$ .*
- (ii) *If  $|y_i| \geq \sqrt{\alpha}$ ,  $i = 0, 1, \dots, k-1$ , then  $\left| \frac{\partial f_k}{\partial y}(\theta, y) \right| \geq C_* \sqrt{\alpha} \sigma^k$ . If, in addition,  $|y_k| \leq \delta_*$ , then  $\left| \frac{\partial f_k}{\partial y}(\theta, y) \right| \geq C_* \sigma^k$ .*

Following [5], in the whole paper, we shall fix

$$\eta = \frac{\log \sigma}{8 \log 32}.$$

This choice of  $\eta$  is only used once for proving the first Claim in Proposition 4.2.

### 3. ADMISSIBLE CURVES

In this section, following previous works, we shall introduce the concept of *admissible curves*, which are images of horizontal curves under iteration of  $F$ . Then we shall study analytical properties of admissible curves and show that they are *nearly horizontal* but *non-flat*.

To begin with, let us give some frequently used notations. Given a curve  $X : I \rightarrow \mathbb{R}$  defined on some interval  $I$ , denote its graph by  $\widehat{X}$ , i.e.  $\widehat{X} = \{(\theta, X(\theta)) : \theta \in I\}$ . Note that  $X$  and  $\widehat{X}$  are determined by each other, and by abusing terminology, both of them will be called curves. For  $h : \widehat{I} \rightarrow \widehat{J}$ , diffeomorphic,  $I \subset \widehat{I}$ ,  $J = h(I)$  and  $X : \widehat{I} \rightarrow \mathbb{R}$ ,  $F(\widehat{X}|_I)$  denotes the graph of the curve defined on  $J$  by  $h(\theta) \mapsto f_1(\theta, X(\theta))$ ,  $\theta \in I$ .

Now we can specify the precise meaning of admissible curves in our situation.

**Definition 3.1** (Admissible Curve). An analytic function  $X$  defined on some  $\omega \in \mathcal{P}_0$  is called an admissible curve, if there exist  $y \in \widehat{I}_b$ ,  $n \geq 1$  and  $\omega_n \in \mathcal{P}_n$ , such that for the horizontal curve  $Y \equiv y$ ,  $\widehat{X} = F^n(\widehat{Y}|_{\omega_n})$ .

*Remark.* By definition, if  $X$  is an admissible curve, then  $F^n(\widehat{X})$  splits into a union of admissible curves for each  $n \in \mathbb{N}$ .

The main result of this section is:

**Proposition 3.1.** *There exist  $l_0 \in \mathbb{N}$  and  $A > B > 0$ , such that when  $\alpha$  is sufficiently small, any admissible curve  $X : \omega \rightarrow \mathbb{R}$  satisfies that*

$$A\alpha \geq \sum_{i=1}^{l_0+1} |X^{(i)}(\theta)| \geq \sum_{i=1}^{l_0} |X^{(i)}(\theta)| \geq B\alpha, \quad \forall \theta \in \omega. \quad (19)$$

To prove Proposition 3.1, the basic idea is to approximate the derivative of admissible curves by functions in the family  $\mathcal{T}$ , which is guaranteed by:

**Lemma 3.2.** *For each  $l \in \mathbb{N}$ , there exist  $C_i > 0$ ,  $i = 0, 1, \dots, l$ , such that the following statement holds when  $\alpha$  is small enough. Let  $\omega \in \mathcal{P}_0$  and let  $X : \omega \rightarrow \widehat{I}_b$  be an admissible curve. Then there exists  $T \in \mathcal{T}_\omega$ , such that*

$$|(X' - \alpha T)^{(i)}(\theta)| \leq C_i \alpha^2, \quad \forall \theta \in \omega, \forall 0 \leq i \leq l. \quad (20)$$

*Moreover, for each  $\omega_1 \in \mathcal{P}_1$  with  $\omega_1 \subset \omega$  and each  $\theta_0 \in h(\omega_1)$ , if we denote  $\widehat{X}_1 = F(\widehat{X}|_{\omega_1})$ ,  $\tau = (h|_{\omega_1})^{-1}$  and*

$$T_1 = (\phi \circ \tau)' + Q'_b(X(\tau\theta_0)) \cdot T \circ \tau \cdot \tau' \quad \text{on } h(\omega_1), \quad (21)$$

*then  $T_1 \in \mathcal{T}_{h(\omega_1)}$  and (20) still holds when  $(X, T, \omega)$  is replaced by  $(X_1, T_1, h(\omega_1))$ .*

*Proof.* By definition,  $X$  is the  $F^n$  image of a constant curve  $Y \equiv y$  for some  $n \geq 1$  and  $y \in \widehat{I}_b$ . When  $n = 1$ ,  $X = \alpha\phi \circ \tau_0 + Q_b(y)$  for some inverse branch  $\tau_0$  of  $h^{-1}$  defined on  $\omega$ . Since  $T := (\phi \circ \tau_0)' \in \mathcal{T}_\omega$  and  $X' = \alpha T$ , (20) holds automatically in this case. To prove the lemma in full generality, by induction on  $n$ , it suffices to prove the following statement: for

each  $l \in \mathbb{N}$ , there exist  $C_i > 0, i = 0, 1, \dots, l$ , such that if  $X' - \alpha T$  satisfies (20), then for  $T_1$  defined in (21),  $T_1 \in \mathcal{T}_{h(\omega_1)}$  and

$$|(X'_1 - \alpha T_1)^{(i)}(\theta)| \leq C_i \alpha^2, \quad \forall \theta \in h(\omega_1), \forall 0 \leq i \leq l. \quad (22)$$

For the  $\theta_0$  appearing in the statement of the lemma, denote  $y_0 = X(\tau\theta_0)$  and  $D = Q'_b(y_0) = -2y_0$ . Since  $|D| \leq 4$ , by the definition of  $\mathcal{T}$ , obviously  $T_1 \in \mathcal{T}_{h(\omega_1)}$ . Because

$$X_1(\theta) = \alpha\phi(\tau\theta) + Q_b(X(\tau\theta)), \quad \forall \theta \in h(\omega_1),$$

it follows that

$$X'_1 - \alpha T_1 = [D \cdot (X' - \alpha T) - 2(X - y_0) \cdot X'] \circ \tau \cdot \tau' \quad \text{on } h(\omega_1). \quad (23)$$

To complete the proof, we only need to show that  $X'_1 - \alpha T_1$  satisfies (22). To this end, let us show the existence of  $C_i$  inductively on  $i$ . Firstly, when  $i = 0$ , by (23),

$$\|X'_1 - \alpha T_1\| \leq |D| \cdot \|X' - \alpha T\| \cdot \|\tau'\| + 2\|X'\|^2 \cdot \|\tau'\|.$$

Here and below  $\|\cdot\|$  denotes the maximum modulus norm of functions. Note that  $|D| \leq 4$ ,  $\|\tau'\| \leq \lambda_g^{-1}$ ,  $\|T\| \leq A_0$  (by (14)) and  $\|X'\| \leq \|X' - \alpha T\| + \alpha\|T\|$ . Therefore,

$$\|X' - \alpha T\| \leq C_0 \alpha^2 \quad \Rightarrow \quad \|X'_1 - \alpha T_1\| \leq \lambda_g^{-1} [4C_0 + 2(C_0\alpha + A_0)^2] \alpha^2.$$

It follows that, if we choose  $C_0$  large, say  $C_0 = \frac{3A_0^2}{\lambda_g^{-4}}$ , then when  $\alpha$  is small,  $X'_1 - \alpha T_1$  satisfies (20) for  $i = 0$ .

Now assume that for some  $1 \leq j \leq l$ ,  $C_0, C_1, \dots, C_{j-1}$  have been chosen, so that  $X'_1 - \alpha T_1$  satisfies (22) for  $0 \leq i \leq j-1$ . Let us determine the choice of  $C_j$ . Differentiating (23)  $j$  times, we obtain that

$$(X'_1 - \alpha T_1)^{(j)} = [D \cdot (X' - \alpha T)^{(j)} - 2(X - y_0) \cdot X^{(j+1)}] \circ \tau \cdot (\tau')^{j+1} + P_j + Q_j. \quad (24)$$

Here  $P_j$  is a linear combination of  $(X' - \alpha T)^{(i)} \circ \tau$ ,  $0 \leq i \leq j-1$ , and  $Q_j$  is a homogeneous quadratic polynomial of  $(X - y_0)^{(i)} \circ \tau$ ,  $0 \leq i \leq j$ . For both  $P_j$  and  $Q_j$ , their coefficients are polynomials of  $\tau^{(i)}$ ,  $1 \leq i \leq j+1$ . By (8),  $\|\tau^{(i)}\| \leq D_{\xi,i}$ . By (14),  $\|T^{(i)}\| \leq A_i$ . Moreover,  $\|(X' - \alpha T)^{(i)}\| \leq C_i \alpha^2$ ,  $i = 0, \dots, j-1$ . Therefore, on the one hand, there exists  $M_j > 0$ , such that

$$\|P_j\| + \|Q_j\| \leq M_j \alpha^2.$$

On the other hand,

$$\|X - y_0\| \leq \|X'\| \leq A_0 \alpha + C_0 \alpha^2 \quad \text{and} \quad \|X^{(j+1)}\| \leq \|(X' - \alpha T)^{(j)}\| + \alpha A_j.$$

Substituting the inequalities above into (24), we can conclude that if  $\|(X' - \alpha T)^{(j)}\| \leq C_j \alpha^2$ , then

$$\|(X'_1 - \alpha T_1)^{(j)}\| \leq \lambda_g^{-j-1} [4C_j + 2(A_0 + C_0\alpha)(A_j + C_j\alpha)] \alpha^2 + M_j \alpha^2.$$

As a result, if we choose  $C_j$  large, say  $C_j = 2M_j + A_0 A_j$ , then when  $\alpha$  is small, (20) holds for  $X'_1 - \alpha T_1$  with  $i = j$ , which completes the induction, and hence the lemma follows.  $\square$

Now we can deduce Proposition 3.1 from Lemma 2.6 and Lemma 3.2.

*Proof of Proposition 3.1.* Let  $l_0, B$  be as in Lemma 2.6, let  $A = 2 \sum_{i=0}^{l_0} A_i$ , where  $A_i$ 's are given in (14), and let  $C_0, C_1, \dots, C_{l_0}$  be determined by Lemma 3.2 for  $l = l_0$ . Assume that  $\alpha$  is so small that  $(l_0 + 1)C_i\alpha < \min\{A/2, B\}$ ,  $i = 0, 1, \dots, l_0$  and that Lemma 3.2 holds for  $l = l_0$ .

By Lemma 3.2, there exists  $T \in \mathcal{T}_\omega$ , such that  $X' - \alpha T$  satisfies (20) for  $l = l_0$ . Due to Lemma 2.6,  $T$  satisfies (14) and (15). Then the choice of constants in the previous paragraph guarantees that (19) holds for  $X$ .  $\square$

As a direct application of the non-flat property of admissible curves, we have the following control of recurrence to  $|y| \leq \alpha$ , which is an analogue of [5, Corollary 5.4].

**Corollary 3.3.** *There exists  $\epsilon_* > 0$ , such that if  $\alpha$  is sufficiently small, then for any admissible curve  $X$  defined on  $\omega \in \mathcal{P}_0$  and any  $0 < \epsilon \leq \epsilon_*$ , we have:*

$$|\{\theta \in \omega : |X(\theta)| \leq \alpha\epsilon\}| \leq \epsilon^{\frac{1}{2l_0}}. \quad (25)$$

*Proof.* By (19), for any  $\theta_0 \in \omega$ , there exists  $1 \leq i \leq l_0$  with  $|X^{(i)}(\theta_0)| \geq B\alpha/l_0$ . Since  $\|X^{(i+1)}\| \leq A\alpha$ , it follows that if  $|\theta - \theta_0| \leq \frac{B}{2Al_0}$ , then  $|X^{(i)}(\theta)| \geq \frac{B\alpha}{2l_0}$ . Therefore, we can divide  $\omega$  into a disjoint union of intervals  $J_j$ ,  $j = 1, 2, \dots, m$  with the following properties:

- $|J_j| \geq \frac{B}{2Al_0}$ ;
- there exists  $1 \leq i_j \leq l_0$ , such that  $|X^{(i_j)}(\theta)| \geq \frac{B\alpha}{2l_0}$ ,  $\forall \theta \in J_j$ .

Then  $m \leq \frac{2Al_0|I_a|}{B}$ , and by [5, Lemma 5.3], for any  $1 \leq j \leq m$  and any  $\epsilon > 0$ ,

$$|\{\theta \in J_j : |X(\theta)| < \alpha\epsilon\}| < 2^{i_j+1} \left(\frac{2l_0\epsilon}{B}\right)^{\frac{1}{i_j}}.$$

Let  $M = \max_{1 \leq j \leq m} 2^{i_j+1} \left(\frac{2l_0}{B}\right)^{\frac{1}{i_j}}$ . Then

$$|\{\theta \in \omega : |X(\theta)| \leq \alpha\epsilon\}| \leq mM\epsilon^{\frac{1}{l_0}} \leq \frac{2Al_0|I_a|}{B} M\epsilon^{\frac{1}{l_0}}, \quad \forall \epsilon \in (0, 1).$$

Choosing  $\epsilon_* \in (0, 1)$  with  $\frac{2Al_0|I_a|}{B} M\epsilon_*^{\frac{1}{2l_0}} \leq 1$ , the conclusion follows.  $\square$

#### 4. CRITICAL RETURN

The crucial ingredient in proving that  $F$  is non-uniformly expanding is to control the approximation of a typical orbit to the critical set  $I_a \times \{0\}$ . For this purpose, we shall show that  $F$  satisfies the so called *slow recurrence conditions* in Proposition 4.3. Following [11, 5], the key step to deduce these conditions is to prove Proposition 4.2, which is an analogue of [11, Lemma 2.6] or [5, Proposition 5.2].

**4.1. A technical proposition.** To begin with, let us prove the following lemma as a substitution of [11, Lemma 2.7] or [5, Lemma 5.5].

**Lemma 4.1.** *There exist  $M_* \in \mathbb{N}$  and  $\epsilon_0 > 0$ , such that the following statement holds when  $\alpha$  is sufficiently small. For each admissible curve  $X$  defined on  $\omega_0 \in \mathcal{P}_0$ , there exist  $\omega^\pm \in \mathcal{P}_{M_*}$  with  $\omega^\pm \subset \omega_0$  and  $h^{M_*}(\omega^+) = h^{M_*}(\omega^-)$ , such that for  $\widehat{Z}^\pm = F^{M_*}(\widehat{X}|_{\omega^\pm})$ ,*

$$\sup_{\theta \in \text{dom}(Z^\pm)} |Z^+(\theta) - Z^-(\theta)| \geq \epsilon_0\alpha. \quad (26)$$

*Proof.* Recall the notations  $\omega_c$  and  $\omega_c^\pm$  introduced just ahead of Lemma 2.6. Since  $h$  is topologically exact on  $I_a$ , there exists  $M_0 \in \mathbb{N}$ , such that  $h^{M_0}(\omega_0) = I_a$  for each  $\omega_0 \in \mathcal{P}_0$ . In particular, given  $\omega_0 \in \mathcal{P}_0$ , there exists  $\omega \in \mathcal{P}_{M_0}$ , such that  $\omega \subset \omega_0$  and  $h^{M_0}(\omega) = \omega_c$ . Denote  $F^{M_0}(\widehat{X}|_\omega)$  by  $\widehat{Y}$  and denote  $D = Q'_b(Y(0))$ . Since  $Y$  is admissible, according to Lemma 3.2, there exists  $T \in \mathcal{T}_{\omega_c}$ , such that  $Y' - \alpha T$  satisfies (20) for  $l = 0$ .

Let  $\widehat{Z}^\pm = F(\widehat{Y}|_{\omega_c^\pm})$ . Then both of  $Z^\pm$  are admissible curves defined on  $h(\omega_c^\pm)$ . Therefore, for  $T^\pm \in \mathcal{T}_{h(\omega_c^\pm)}$  defined in (16) with  $T$  and  $D$  given in the previous paragraph, by applying Lemma 3.2 again, one can see that  $(Z^\pm)' - \alpha T^\pm$  also satisfy (20) for  $l = 0$ . According to (iii) of Lemma 2.6 and the compactness of  $\mathcal{T}_{h(\omega_c^\pm)}$ , there exists  $\epsilon_1 > 0$ , independent of  $T^\pm$ , such that

$$\sup_{\theta \in h(\omega_c^\pm)} |T^+(\theta) - T^-(\theta)| \geq 2\epsilon_1.$$

By (20),  $\|(Z^\pm)' - \alpha T^\pm\| \leq C_0 \alpha^2$ . Therefore, when  $\alpha$  is sufficiently small, it follows that

$$\sup_{\theta \in h(\omega_c^\pm)} |(Z^+)'(\theta) - (Z^-)'(\theta)| \geq \epsilon_1 \alpha.$$

Due to (19),  $|(Z^\pm)''| \leq A\alpha$  on  $h(\omega_c^\pm)$ . Then it is easy to see that (26) holds for  $M_* = M_0 + 1$  and suitable choice of  $\epsilon_0$ .  $\square$

The proposition below is the substitution of [11, Lemma 2.6] or [5, Proposition 5.2] with the same idea of proof. We shall follow the proof in [5]. The main change of ingredients here is to replace [5, Corollary 5.4] and [5, Lemma 5.5] with Corollary 3.3 and Lemma 4.1 in this paper respectively.

Denote

$$M_\alpha = \left\lceil \frac{\log \frac{1}{\alpha}}{\log 32} \right\rceil \quad \text{and} \quad r_0(\alpha) = \left( \frac{1}{2} - 2\eta \right) \log \frac{1}{\alpha}.$$

Note that for  $N_\alpha$  introduced in Lemma 2.7, we have  $M_\alpha \leq \frac{2}{5}N_\alpha$ .

**Proposition 4.2.** *There exists  $\beta_0 > 0$ , such that when  $\alpha$  is sufficiently small, for each admissible curve  $Y$  defined on  $\omega_0 \in \mathcal{P}_0$  and each  $r \geq r_0(\alpha)$ , we have*

$$|\{\theta \in \omega_0 : |f_{M_\alpha}(\theta, Y(\theta))| \leq \sqrt{\alpha} e^{-r}\}| \leq e^{-\beta_0 r}. \quad (27)$$

*Proof.* When  $r \geq \left(\frac{1}{2} + 2\eta\right) \log \frac{1}{\alpha}$ ,  $\sqrt{\alpha} e^{-r} \leq \alpha^{1+2\eta}$ . Since  $F^{M_\alpha}(\widehat{Y})$  splits into a union of admissible curves, the conclusion follows from Corollary 3.3 and Lemma 2.4 immediately by choosing  $\beta_0$  appropriately. Otherwise, it suffices to consider the case  $r = r_0(\alpha)$  and let us follow the argument in [5]. Without loss of generality, we can assume that there exists  $z_0 = (\theta_0, y_0)$  on  $\widehat{Y}$ , such that  $|f_{M_\alpha}(z_0)| \leq \sqrt{\alpha}$ . For each  $i \geq 0$ , denote  $z_i = F^i(z_0)$ . Then by (ii) of Lemma 2.7,

$$\left| \frac{\partial f_{M_\alpha-i}}{\partial y}(z_i) \right| \geq C_* \sigma^{M(\alpha)-i}, \quad 0 \leq i < M_\alpha. \quad (28)$$

Let us summarize some basic estimates of distortion in [5] with slight modification. Let  $y_i = Q_b^i(y_0)$ ,  $i \geq 0$ . Recall the constant  $A$  in (19) and let  $L = \max\{1, A|I_a|\}$ . Denote  $B_i = [y_i - 5^i L \alpha, y_i + 5^i L \alpha]$ ,  $i \geq 0$ . Then

- $Q_b(B_i) \subset B_{i+1}$ ,  $\forall i \geq 0$ ;
- $f_i(\widehat{Y}) \subset B_i$ ,  $\forall i \geq 0$ ;
- $B_i \cap [-\sqrt{\alpha}, \sqrt{\alpha}] = \emptyset$ ,  $0 \leq i < M_\alpha$ .

It follows that for  $0 \leq i < j < M_\alpha$ ,

$$\sum_{k=i}^j \sup_{y, y' \in B_k} |\log |Q'_b(y)| - \log |Q'_b(y')|| \leq \log 2. \quad (29)$$

Following [5], let us define some notations and constants. Firstly, let  $\bar{\sigma} = \sqrt{\sigma}$  and introduce

$$\lambda_j = \left| \frac{\partial f_{M_\alpha-j}}{\partial y}(z_j) \right| / \bar{\sigma}^{M_\alpha-j} \geq C_* \bar{\sigma}^{M(\alpha)-j}, \quad 0 \leq j \leq M_\alpha. \quad (30)$$

Then obviously  $\lambda_j / \lambda_{j+1} = |Q'_b(f_j(z_0))| / \bar{\sigma} < 4$ ,  $0 \leq j < M_\alpha - 1$ . Secondly, recalling the constant  $M_*$  appearing in Lemma 4.1, let  $\kappa > 4^{M_*}$  be a constant satisfying that

$$\kappa \left( 4^{-M_*-1} \kappa \epsilon_0 - 2A|I_a| - 8(1 - \bar{\sigma}^{-1})^{-1} \right) \geq 3.$$

Thirdly, let  $0 = t_1 < t_2 < \dots < t_q \leq M_\alpha$  be all the times such that

$$t_{i+1} := \max \{ s : t_i < s \leq M_\alpha, \lambda_{t_i} \leq 2\kappa \lambda_s \}.$$

By definition,

- $\lambda_{t_q} \leq 2\kappa$ ;
- $\lambda_j < \lambda_{t_{i+1}}$  when  $t_{i+1} < j < M_\alpha$ ;
- $\frac{\kappa}{2} < \lambda_{t_i} / \lambda_{t_{i+1}} < 4^{t_{i+1}-t_i} \Rightarrow t_{i+1} \geq t_i + M_*$ .

Finally, let

$$k_0(\alpha) := \max \left\{ 1 \leq i \leq q : \lambda_{t_i} \geq 2\kappa e^{-r_0(\alpha)} / \sqrt{\alpha} \right\}.$$

Following Claim 1 in the proof of [5, Proposition 5.2], we have:

**Claim.** *When  $\alpha$  is sufficiently small,*

$$k_0(\alpha) \geq \eta r_0(\alpha) / \log(2\kappa). \quad (31)$$

*Proof of Claim.* By definition,  $\lambda_{t_i} \leq 2\kappa \lambda_{t_{i+1}}$ ,  $i = 0, 1, \dots, q-1$ . Because  $\lambda_{t_q} \leq 2\kappa$ , by the choice of  $k_0(\alpha)$ ,  $k_0(\alpha) < q$  and hence

$$2\kappa e^{-r_0(\alpha)} / \sqrt{\alpha} \geq \lambda_{t_{k_0(\alpha)+1}} \geq (2\kappa)^{-k_0(\alpha)} \lambda_{t_1} \geq C_* \cdot (2\kappa)^{-k_0(\alpha)} \cdot \bar{\sigma}^{M_\alpha}.$$

Recall that  $r_0(\alpha) = (\frac{1}{2} - 2\eta) \log(1/\alpha)$ ,  $M_\alpha = [-\frac{\log \alpha}{\log 32}]$ ,  $\bar{\sigma} = \sqrt{\sigma}$  and  $\eta = \frac{\log \sigma}{8 \log 32}$ . The conclusion follows.  $\square$

Denote

$$E := \{ \theta \in \omega_0 : |f_{M_\alpha}(\theta, Y(\theta))| \leq \sqrt{\alpha} e^{-r_0(\alpha)} \}.$$

Then our aim is to prove that there exists  $\beta_0 > 0$ , independent of  $Y$ , such that when  $\alpha$  is sufficiently small,  $|E| \leq e^{-\beta_0 r_0(\alpha)}$ . For each  $1 \leq i \leq k_0(\alpha)$ , define

$$\Omega_i := \{ \omega \in \mathcal{P}_{t_i} : \bar{\omega} \cap E \neq \emptyset \} \quad \text{and} \quad E_i := \cup_{\omega \in \Omega_i} \bar{\omega}.$$

By definition,  $\cap_{i=0}^{k_0(\alpha)} E_i \supset E$ .

**Claim.** *For each  $1 \leq i < k_0(\alpha)$  and each  $\omega_i \in \Omega_i$ , there exists  $\omega \in \mathcal{P}_{t_{i+1}}$ , such that  $\omega \subset \omega_i$  and  $\bar{\omega} \cap E = \emptyset$ .*

*Proof of Claim.* Given  $\omega_i \in \Omega_i$ ,  $\widehat{Y}_i := F^{t_i}(\widehat{Y}|_{\omega_i})$  is an admissible curve. According to Lemma 4.1, there exist  $\omega^\pm \in \mathcal{P}_{M_*}$  with  $\omega^\pm \subset h^{t_i}(\omega_i)$  and  $h^{M_*}(\omega^+) = h^{M_*}(\omega^-) := \widehat{J}$ , such that for  $\widehat{Z}^\pm = F^{M_*}(\widehat{Y}_i|_{\omega^\pm})$ , (26) holds. Since  $t_i + M_* \leq t_{i+1}$ , in particular, there exist  $\omega_{i+1}^\pm \in \mathcal{P}_{t_{i+1}}$ , such that  $\omega_{i+1}^\pm \subset \omega_i$ ,  $h^{t_i+M_*}(\omega_{i+1}^+) = h^{t_i+M_*}(\omega_{i+1}^-) := J \subset \widehat{J}$ , and

$$\sup_{\theta \in J} |Z^+(\theta) - Z^-(\theta)| \geq \epsilon_0 \alpha. \quad (32)$$

Denote  $t_{i+1} - t_i - M_*$  by  $n$  temporarily. By definition, given  $\theta \in \text{dom}(Y_{i+1}^\pm)$ , there exists  $\theta_0 \in J$ , such that  $h^n(\theta_0) = \theta$ . Due to (29),

$$|Y_{i+1}^+(\theta) - Y_{i+1}^-(\theta)| = |f_n(\theta_0, Z^+(\theta_0)) - f_n(\theta_0, Z^-(\theta_0))| \geq \frac{1}{2} \left| \frac{\partial f_n}{\partial y}(z_{t_i+M_*}) \right| \times |Z^+(\theta_0) - Z^-(\theta_0)|,$$

where

$$\left| \frac{\partial f_n}{\partial y}(z_{t_i+M_*}) \right| \geq 4^{-M_*} \left| \frac{\partial f_{t_{i+1}-t_i}}{\partial y}(z_{t_i}) \right| = \frac{\lambda_{t_i}}{4^{M_*} \lambda_{t_{i+1}}} \times \bar{\sigma}^{t_{i+1}-t_i}.$$

Then (32) and the two inequalities above imply that

$$\sup_{\theta \in \text{dom}(Y_{i+1}^\pm)} |Y_{i+1}^+(\theta) - Y_{i+1}^-(\theta)| \geq \frac{1}{2} \frac{\lambda_{t_i}}{4^{M_*} \lambda_{t_{i+1}}} \epsilon_0 \alpha. \quad (33)$$

For  $t_{i+1} \leq j \leq M_\alpha$ , define

$$\Delta_j := \inf_{\theta^\pm \in \omega_{i+1}^\pm} |f_j(\theta^+, Y(\theta^+)) - f_j(\theta^-, Y(\theta^-))|.$$

By (33) and noting that  $|(Y_{i+1}^\pm)'| \leq A\alpha$ , we have:

$$\Delta_{t_{i+1}} = \inf_{\theta^\pm \in \text{dom}(Y_{i+1}^\pm)} |Y_{i+1}^+(\theta^+) - Y_{i+1}^-(\theta^-)| \geq \frac{1}{2} \frac{\lambda_{t_i}}{4^{M_*} \lambda_{t_{i+1}}} \epsilon_0 \alpha - 2A|I_a|\alpha \geq (4^{-M_*-1} \kappa \epsilon_0 - 2A|I_a|) \alpha. \quad (34)$$

As in the proof of [5, Proposition 5.2], denote  $D_j = \min_{y \in B_j} |Q'_b(y)|$ . Since

$$\Delta_{j+1} \geq D_j \Delta_j - 2\alpha,$$

by induction,

$$\Delta_{M_\alpha} \geq \Delta_{t_{i+1}} \prod_{j=t_{i+1}}^{M_\alpha-1} D_j - 2\alpha \left( 1 + \sum_{j=t_{i+1}+1}^{M_\alpha-1} \left( \prod_{l=j}^{M_\alpha-1} D_l \right) \right).$$

By (29) and (30),

$$\lambda_j \bar{\sigma}^{M_\alpha-j} / 2 \leq \prod_{l=j}^{M_\alpha-1} D_l \leq 2\lambda_j \bar{\sigma}^{M_\alpha-j}.$$

Noting that  $\lambda_j < \lambda_{t_{i+1}}$  when  $j > t_{i+1}$ , combing the two inequalities above, it is easy to obtain that

$$\Delta_{M_\alpha} \geq \lambda_{t_{i+1}} \bar{\sigma}^{M_\alpha-t_{i+1}} (\Delta_{t_{i+1}} / 2 - 4\alpha(1 - \bar{\sigma}^{-1})^{-1}).$$

Substituting (34) into the inequality above and making use of  $\lambda_{t_{i+1}} \geq 2\kappa e^{-r_0(\alpha)} / \sqrt{\alpha}$  and the choice of  $\kappa$ , we have:

$$\Delta_{M_\alpha} \geq 3e^{-r_0(\alpha)} \sqrt{\alpha}.$$



That is to say,  $\overline{\omega_{i+1}^\pm}$  cannot both intersect  $E$ , i.e. the claim holds.  $\square$

Now let us return to the proof of the proposition. Let  $N \geq 2M_\alpha/k_0(\alpha)$  be a large integer independent of  $\alpha$ . Then

$$\#\{1 \leq i < k_0(\alpha) : t_{i+1} - t_i \leq N\} \geq \frac{k_0(\alpha)}{2}.$$

By Lemma 2.4, there exists  $p > 0$ , only depending on  $N$ , such that for each pair  $\omega' \subset \omega$  with  $\omega \in \mathcal{P}_n$ ,  $\omega' \in \mathcal{P}_{n'}$ ,  $n < n' \leq n + N$ , we have  $|\omega'| \geq p|\omega|$ . By the second claim, it follows that when  $t_{i+1} \leq t_i + N$ ,  $|E_{i+1}| \leq (1-p)|E_i|$ . Therefore,

$$|E| \leq (1-p)^{k_0(\alpha)/2}.$$

Then by (31),  $|E| \leq e^{-\beta_0 r_0(\alpha)}$  for some constant  $\beta_0 > 0$ , which completes the proof.  $\square$

**4.2. Slow recurrence conditions.** To make the argument slightly simpler, let us adopt the following weakened forms of Corollary 3.3 and Proposition 4.2. There exists  $0 < \beta \leq \min\{\frac{1}{2\beta_0}, \beta_0\}$ , such that when  $\alpha$  is small enough, for each admissible curve  $X : \omega \rightarrow \mathbb{R}$ , we have:

- If  $\epsilon \leq \alpha^2$ , then

$$|\{\theta \in \omega : |X(\theta)| \leq \epsilon\}| \leq \epsilon^\beta. \quad (35)$$

- If  $\epsilon \leq \alpha^{1-2\eta}$ , then

$$|\{\theta \in \omega : |f_{M_\alpha}(\theta, X(\theta))| \leq \epsilon\}| \leq \epsilon^\beta. \quad (36)$$

Based on (35) and (36), following the “Large deviations” argument in [11], we can deduce the following version of slow recurrence conditions.

**Proposition 4.3.** *There exists  $c > 0$ , such that when  $\alpha$  is sufficiently small, the following statement holds. For each  $\epsilon > 0$ , there exists  $\tilde{\delta} = \tilde{\delta}(\epsilon) \in (0, \frac{1}{2})$ , independent of  $\alpha$ , such that when  $n \in \mathbb{N}$  is sufficiently large,*

$$\left| \left\{ (\theta, y) \in I_a \times \widehat{I}_b : \sum_{\substack{0 \leq i < n \\ |f_i(\theta, y)| < \delta}} \log \frac{1}{|f_i(\theta, y)|} > \epsilon n \right\} \right| \leq e^{-c\epsilon\beta\sqrt{n}}, \quad (37)$$

where  $\delta = \tilde{\delta}\alpha^{1-2\eta}$ . In particular, for Lebesgue a.e.  $(\theta, y) \in I_a \times \widehat{I}_b$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{\substack{0 \leq i < n \\ |f_i(\theta, y)| < \delta}} \log \frac{1}{|f_i(\theta, y)|} \leq \epsilon. \quad (38)$$

*Remark.* It is well known that, starting from the estimate shown in (37), kinds of statistical properties beyond the Main Theorem can be obtained. See, for example [1], for a survey on this topic.

To begin with the proof of Proposition 4.3, for each  $0 < \tilde{\delta} < \frac{1}{2}$ , let  $\delta = \tilde{\delta}\alpha^{1-2\eta}$ , and denote

$$\Delta = \Delta(\delta) := \left\lceil \frac{\log \frac{1}{\delta}}{\log \lambda_g} \right\rceil.$$

Fix an arbitrary  $y_0 \in \widehat{I}_b \setminus \{0\}$ . Let

$$\mathcal{R} = \mathcal{R}(y_0) := \{\theta \in I_a : h^n(\theta) \notin u(PC_a) \text{ and } f_n(\theta, y_0) \neq 0, \forall n \geq 0\}.$$

Note that  $\mathcal{R}$  is of full Lebesgue measure in  $I_a$ . For each  $k \geq 0$  and each  $\theta \in I_a$ , if  $\theta \in \mathcal{R}$  and  $|f_k(\theta, y_0)| < \delta$ , let  $q_k(\theta)$  be the unique positive integer satisfying

$$\lambda_g^{-q_k(\theta)-1} < |f_k(\theta, y_0)| \leq \lambda_g^{-q_k(\theta)};$$

otherwise, let  $q_k(\theta) = 0$ . By definition, either  $q_k(\theta) = 0$  or  $q_k(\theta) \geq \Delta$ .

Fix  $\varepsilon > 0$ . For  $K \in \mathbb{N}$ , let

$$E_K = E_K(\varepsilon, \tilde{\delta}, y_0) := \left\{ \theta \in \mathcal{R} : \sum_{k=1}^K q_k(\theta) \geq \varepsilon K \right\}.$$

By Fubini's theorem, Proposition 4.3 can be reduced to the following lemma.

**Lemma 4.4.** *When  $\alpha$  is small enough, the following statement holds. Given  $\varepsilon > 0$ , there exist  $0 < \tilde{\delta} < \frac{1}{2}$ , independent of  $\alpha$  or  $y_0$ , and  $K_0 \in \mathbb{N}$ , independent of  $y_0$ , such that when  $K > K_0$ ,*

$$|E_K| \leq \lambda_g^{-\frac{\varepsilon\beta}{10}\sqrt{K}}.$$

*Proof.* Let  $Y$  denote the constant curve  $Y \equiv y_0$ . We shall make use of the admissibility of pieces of  $F^n(\widehat{Y})$  for appropriate  $n$  repeatedly.

Given  $K \in \mathbb{N}$ , let

$$E_K^2 := \left\{ \theta \in E_K : \exists 1 \leq k \leq K, q_k(\theta) > \sqrt{K} \right\} \quad \text{and} \quad E_K^1 := E_K \setminus E_K^2.$$

By definition,

$$E_K^2 \subset \bigcup_{k=1}^K \bigcup_{\omega \in \mathcal{P}_k} \left\{ \theta \in \mathcal{R} \cap \omega : q_k(\theta) > \sqrt{K} \right\}.$$

For each  $k \geq 1$  and each  $\omega \in \mathcal{P}_k$ ,  $X_\omega := F^k(\widehat{Y})|_\omega$  is an admissible curve, and

$$h^k(\{\theta \in \mathcal{R} \cap \omega : q_k(\theta) > \sqrt{K}\}) \subset \{\theta \in h^k(\omega) : |X_\omega(\theta)| < \lambda_g^{-\sqrt{K}}\}.$$

Therefore, when  $K$  is large, say  $K > 4\left(\frac{\log \alpha}{\log \lambda_g}\right)^2$ , by (35) together with (9),

$$|\{\theta \in \mathcal{R} \cap \omega : q_k(\theta) > \sqrt{K}\}| \leq C_d \lambda_g^{-\beta\sqrt{K}} |\omega|.$$

It follows that, when  $K$  is large,

$$|E_K^2| \leq C_d |I_a| \cdot K \lambda_g^{-\beta\sqrt{K}} \leq \lambda_g^{-\frac{\beta}{2}\sqrt{K}} \leq \lambda_g^{-\frac{\varepsilon\beta}{2}\sqrt{K}}. \quad (39)$$

Given  $K \geq 4M_\alpha^2/\varepsilon^2$ , to estimate the size of  $E_K^1$ , let us denote  $L := 2[\sqrt{K}]$ , and for every  $0 \leq p \leq L-1$ , define

$$\mathcal{M}_p := \{M_\alpha \leq k \leq K : k \equiv p \pmod{L}\}$$

and

$$E_{K,p}^1 := \left\{ \theta \in E_K^1 : \sum_{k \in \mathcal{M}_p} q_k(\theta) \geq \frac{\varepsilon K}{2L} \right\}.$$

By definition, if  $\theta \in E_K^1$  with  $K \geq 4M_\alpha^2/\varepsilon^2$ , then  $q_k(\theta) \leq \sqrt{K}$ ,  $\forall 1 \leq k \leq K$ , and hence

$$\sum_{p=0}^{L-1} \sum_{k \in \mathcal{M}_p} q_k(\theta) = \sum_{k=M_\alpha}^K q_k(\theta) > \varepsilon K - M_\alpha \sqrt{K} \geq \frac{\varepsilon K}{2}.$$

It implies that  $E_K^1 \subset \bigcup_{p=0}^{L-1} E_{K,p}^1$ .

Now let us fix  $K$  and  $p$  and estimate  $|E_{K,p}^1|$ . For each  $\mathbf{r} = (r_k)_{k \in \mathcal{M}_p}$ , denote  $\|\mathbf{r}\| = \sum_k r_k$  and let

$$E_{K,p}^1(\mathbf{r}) := \left\{ \theta \in E_{K,p}^1 : q_k(\theta) = r_k \text{ for all } k \in \mathcal{M}_p \right\}.$$

**Claim.** *Provided that  $\alpha$  is small enough, for each  $\mathbf{r} \in \{0, 1, \dots, \lfloor \sqrt{K} \rfloor\}^{\mathcal{M}_p}$ , when  $K$  is large (independent of  $y_0$ ), we have*

$$|E_{K,p}^1(\mathbf{r})| \leq \lambda_g^{-\frac{2\beta}{3}\|\mathbf{r}\|}.$$

*Proof of Claim.* When  $\|\mathbf{r}\| = 0$ , there is nothing to prove. Otherwise, let  $M_\alpha \leq k_1 < k_2 < \dots < k_l$  be all the elements in  $\mathcal{M}_p$  with  $r_{k_j} > 0$ . For each  $1 \leq j \leq l$  and each  $\theta \in E_{K,p}^1(\mathbf{r})$ , let  $J'_j(\theta)$  be the element in  $\mathcal{P}_{k_j-M_\alpha}$  containing  $\theta$ , and let  $J_j(\theta)$  be the element in  $\mathcal{P}_{k_j+r_{k_j}}$  containing  $\theta$ . Finally, let

$$\Omega'_j = \bigcup_{\theta \in E_{K,p}^1(\mathbf{r})} J'_j(\theta) \quad \text{and} \quad \Omega_j = \bigcup_{\theta \in E_{K,p}^1(\mathbf{r})} J_j(\theta).$$

Since  $k_{j+1} - k_j \geq L \geq r_{k_j} + M_\alpha$ , any element of  $\mathcal{P}_{k_{j+1}-M_\alpha}$  is either contained in  $\Omega_j$  or disjoint from  $\Omega_j$ , which implies that  $\Omega'_{j+1} \subset \Omega_j$ .

Let us show that for each  $j = 1, 2, \dots, l$  and each component  $J'_j$  of  $\Omega'_j$ ,

$$|\Omega_j \cap J'_j| \leq 2^\beta C_d \cdot \lambda_g^{-\beta r_{k_j}} |J'_j|. \quad (40)$$

Indeed, for each  $\theta' \in \Omega_j \cap J'_j$ , there exists  $\theta \in E_{K,p}^1(\mathbf{r})$  such that  $\theta' \in J_j(\theta)$ . Because  $|h'| \geq \lambda_g$  on  $I_a$  and  $h^{k_j}(J_j(\theta)) \in \mathcal{P}_{r_{k_j}}$ ,  $|h^{k_j}(\theta) - h^{k_j}(\theta')| \leq \lambda_g^{-r_{k_j}} |I_a|$ . Since  $\widehat{Z} = F^{k_j}(\widehat{Y}|_{J_j(\theta)})$  is a piece of admissible curve and since  $q_{k_j}(\theta) = r_{k_j}$ , it follows that

$$|Z(h^{k_j}(\theta'))| \leq |Z(h^{k_j}(\theta')) - Z(h^{k_j}(\theta))| + |Z(h^{k_j}(\theta))| \leq (A|I_a| \cdot \alpha + 1) \lambda_g^{-r_{k_j}} \leq 2\lambda_g^{-r_{k_j}}.$$

Denote  $\widehat{X} := F^{k_j-M_\alpha}(\widehat{Y}|_{J'_j})$ . The inequality above implies that

$$|f_{M_\alpha}(\theta, X(\theta))| \leq 2\lambda_g^{-r_{k_j}}, \quad \forall \theta \in h^{k_j-M_\alpha}(\Omega_j \cap J'_j).$$

Since  $r_{k_j} \geq \Delta$ ,  $2\lambda_g^{-r_{k_j}} \leq 2\delta \leq \alpha^{1-2\eta}$ . Then by (36),

$$|h^{k_j-M_\alpha}(\Omega_j \cap J'_j)| \leq 2^\beta \lambda_g^{-\beta r_{k_j}}.$$

Due to (9), the estimate (40) follows. As a result,

$$|\Omega_j| \leq 2^\beta C_d \cdot \lambda_g^{-\beta r_{k_j}} |\Omega'_j|.$$

Therefore, an obvious induction on  $j$  implies that

$$|E_{K,p}^1(\mathbf{r})| \leq |\Omega_l| \leq 2^{\beta l} C_d^l \cdot \lambda_g^{-\sum_{j=1}^l \beta r_{k_j}} |\Omega'_1| \leq \lambda_g^{-\frac{2\beta}{3}\|\mathbf{r}\|},$$

provided that  $\alpha$  is small, and  $\|\mathbf{r}\| \geq \Delta$  is large accordingly.  $\square$

Now let us estimate the number of possible constrained configurations of  $\mathbf{r}$ . Given  $R \in \mathbb{N}$ , define

$$Q(K, p, R) := \# \left\{ \mathbf{r} \in \{0, 1, \dots, \lfloor \sqrt{K} \rfloor\}^{M_p} : \|\mathbf{r}\| = R \text{ and } E_{K,p}^1(\mathbf{r}) \neq \emptyset \right\}.$$

**Claim.** *Given  $\varepsilon > 0$ , there exists  $0 < \tilde{\delta} < \frac{1}{2}$ , such that when  $K$  is large (independent of  $y_0$ ),*

$$Q(K, p, R) \leq \lambda_g^{\frac{\beta}{6}R}, \quad \forall R \geq \frac{\varepsilon K}{2L}.$$

*Proof of Claim.* Since each nonzero  $r_k$  is no less than  $\Delta$ ,

$$Q(K, p, R) \leq \sum_{i=1}^{\min(\#\mathcal{M}_p, \lfloor R/\Delta \rfloor)} \binom{\#\mathcal{M}_p}{i} \binom{R - i(\Delta - 1) - 1}{i - 1},$$

where the first binomial coefficient counts the position of nonzero  $r_k$ 's, and the second counts the distribution of their values.

By Stirling's formula,  $R^{-1} \log \binom{R}{\lfloor R/\Delta \rfloor} \rightarrow 0$  uniformly in  $R$  as  $\Delta \rightarrow \infty$ . Therefore, by the definition of  $\Delta$ , when  $\tilde{\delta}$  is sufficiently small, for  $1 \leq i \leq \lfloor R/\Delta \rfloor$ , we have:

$$\binom{R - i(\Delta - 1) - 1}{i - 1} \leq \binom{R}{\lfloor R/\Delta \rfloor} \leq \lambda_g^{\frac{\beta}{12}R}.$$

Since  $\#\mathcal{M}_p \approx \frac{\sqrt{K}}{2}$ , when  $R > \sqrt{\Delta K}$ ,

$$Q(K, p, R) \leq 2^{\#\mathcal{M}_p} \lambda_g^{\frac{\beta}{12}R} \leq \lambda_g^{\frac{\beta}{6}R}.$$

Otherwise, when  $\varepsilon \sqrt{K} \leq R \leq \sqrt{\Delta K}$ ,

$$Q(K, p, R) \leq \lambda_g^{\frac{\beta}{12}R} \sum_{i=1}^{\lfloor R/\Delta \rfloor} \binom{\#\mathcal{M}_p}{i} \leq \lfloor R/\Delta \rfloor \lambda_g^{\frac{\beta}{12}R} \binom{\#\mathcal{M}_p}{\lfloor R/\Delta \rfloor}.$$

Noting that  $\#\mathcal{M}_p < R/\varepsilon$ , in this case the conclusion also follows from Stirling's formula, provided that  $\tilde{\delta}$  is small, and  $\Delta/\varepsilon$  is large accordingly.  $\square$

By combining the last two claims, we have:

$$|E_{K,p}^1| \leq \sum_{\mathbf{r} \in \{0, 1, \dots, \sqrt{K}\}^{M_p}} |E_{K,p}^1(\mathbf{r})| \leq \sum_{R \geq \frac{\varepsilon K}{2L}} \lambda_g^{-\frac{\beta}{2}R} \leq (1 - \lambda_g^{-\frac{\beta}{2}})^{-1} \lambda_g^{-\frac{\varepsilon\beta}{8}} \sqrt{K}.$$

It follows that, for  $K$  large,

$$|E_K^1| \leq \sum_{p=0}^{L-1} |E_{K,p}^1| \leq L(1 - \lambda_g^{-\frac{\beta}{2}})^{-1} \lambda_g^{-\frac{\varepsilon\beta}{8}} \sqrt{K} \leq \lambda_g^{-\frac{\varepsilon\beta}{9}} \sqrt{K},$$

which, together with (39), completes the proof of the lemma.  $\square$

## 5. PROOF OF THE MAIN THEOREM

To prove the Main Theorem, it suffices to prove the same statements for  $F$  instead of  $\mathcal{F}$ . Recall that, when  $\alpha$  is small enough,  $F$  maps  $I_a \times \widehat{I}_b$  into itself. Apparently the interesting dynamics of  $F$  is concentrated on the invariant set

$$\Lambda = \bigcap_{n=0}^{\infty} F^n(I_a \times \widehat{I}_b),$$

because for  $(\theta, y)$  outside  $\Lambda$ ,  $f_n(\theta, y) \rightarrow \infty$  as  $n \rightarrow \infty$ .

Let us follow some terminology in [3]. To start with, note that the base dynamics  $h : I_a \rightarrow I_a$  is a  $C^2$  local diffeomorphism outside a finite set  $C_\theta$  of singularities. Let

$$\mathcal{S}_\theta = C_\theta \times \widehat{I}_b \quad \text{and} \quad \mathcal{S}_y = I_a \times \{0\}.$$

Then  $F$  is a  $C^2$  local diffeomorphism outside  $\mathcal{S} = \mathcal{S}_\theta \cup \mathcal{S}_y$ , the so called *singular set* in [3], and the conditions (S1)-(S3) in [3] hold for  $\beta = 1$ . The following subsections are devoted to showing that  $F$  is *nonuniformly expanding* in the sense of [3].

**5.1. Positive Lyapunov exponents.** Using the same argument as in [11], it is easy to deduce vertical positive Lyapunov exponent from Proposition 4.3. We only need to consider the vertical Lyapunov exponent at the point  $(\theta, y)$  where (38) holds.

Denote the  $F$ -orbit of  $(\theta, y)$  by  $\{(\theta_i, y_i)\}_{i \geq 0}$ . Given  $n \in \mathbb{N}$ , let  $0 \leq \nu_1 < \nu_2 < \dots < \nu_s \leq n$  be all the times  $i$  such that  $|f_i(\theta, y)| \leq \sqrt{\alpha}$ . According to Lemma 2.7, we have:

- $\nu_{i+1} - \nu_i \geq N_\alpha$ ,  $1 \leq i < s$ , and in particular  $n \geq (s-1)N_\alpha$ ;
- $\left| \frac{\partial f_{N_\alpha}}{\partial y}(\theta_{\nu_i}, y_{\nu_i}) \right| \geq |y_{\nu_i}| \alpha^{-1+\eta}$ ,  $1 \leq i < s$ ;
- $\left| \frac{\partial f_{\nu_{i+1}-\nu_i-N_\alpha}}{\partial y}(\theta_{\nu_i+N_\alpha}, y_{\nu_i+N_\alpha}) \right| \geq C_* \sigma^{\nu_{i+1}-\nu_i-N_\alpha}$ ,  $1 \leq i < s$ ;
- $\left| \frac{\partial f_{\nu_1}}{\partial y}(\theta_0, y_0) \right| \geq C_* \sigma^{\nu_1}$  and  $\left| \frac{\partial f_{n-\nu_s}}{\partial y}(\theta_{\nu_s}, y_{\nu_s}) \right| \geq C_* |y_{\nu_s}| \sqrt{\alpha} \sigma^{n-\nu_s}$ .

For  $\varepsilon > 0$ , let  $0 < \tilde{\delta} < \frac{1}{2}$  be determined in Proposition 4.3. Then by (38), when  $n$  is sufficiently large, we have:

$$\prod_{i=1}^s |y_{\nu_i}| \geq \delta^s \prod_{\substack{1 \leq i \leq s \\ |y_{\nu_i}| < \delta}} |y_{\nu_i}| \geq \tilde{\delta}^s \alpha^{(1-2\eta)s} e^{-2\varepsilon n}.$$

Therefore,

$$\left| \frac{\partial f_n}{\partial y}(\theta, y) \right| \geq C_*^{s+1} \tilde{\delta}^s \alpha^{\frac{3}{2}-\eta(s+1)} \sigma^{n-(s-1)N_\alpha} e^{-2\varepsilon n}.$$

Since  $\sigma^{N_\alpha} \leq \alpha^{-1}$ ,  $\alpha^{-\eta(s+1)} \sigma^{n-(s-1)N_\alpha} \geq \sigma^{\eta n}$ . Choose  $\varepsilon = \frac{\eta}{6} \log \sigma$ . Note that  $\tilde{\delta}$  is determined by  $\varepsilon$  and  $(s-1)N_\alpha \leq n$ , when  $\alpha$  is small enough,  $\left| \frac{\partial f_n}{\partial y}(\theta, y) \right| \geq \sigma^{\frac{\eta}{2}n}$  for  $n$  large. Then we have:

**Proposition 5.1.** *When  $\alpha$  is sufficiently small, for Lebesgue almost every  $(\theta, y) \in I_a \times \widehat{I}_b$ ,*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \left| \frac{\partial f_n}{\partial y}(\theta, y) \right| \geq \frac{\eta}{2} \log \sigma > 0. \quad (41)$$

Therefore, the vertical Lyapunov exponent is positive. Since the base map  $h$  is uniformly expanding, we have proved that  $F$  has two positive Lyapunov exponents.

**5.2. Existence of a.c.i.p.** We shall apply the results in [3, Theorem C] to obtain the existence of an a.c.i.p. for  $F$ . By definition,

$$DF(\theta, y) = \begin{pmatrix} h'(\theta) & 0 \\ \alpha\phi'(\theta) & Q'_b(y) \end{pmatrix}.$$

Recall that  $|h'(\theta)| \geq \lambda_g > 4 > |Q'_b(y)|$  on  $I_a \times \widehat{I}_b$ , and hence

$$\|DF(\theta, y)^{-1}\| \leq (1 + C\alpha)|Q'_b(y)|^{-1},$$

where  $C > 0$  is a constant independent of  $\alpha$ . Therefore,

$$\frac{1}{n} \sum_{i=0}^{n-1} \log \|DF(F^i(\theta, y))^{-1}\| \leq C\alpha - \frac{1}{n} \log \left| \frac{\partial f_n}{\partial y}(\theta, y) \right|.$$

By (41), for a.e.  $(\theta, y)$  we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|DF(F^i(\theta, y))^{-1}\| \leq -\frac{\eta}{3} \log \sigma < 0,$$

provided that  $\alpha$  is small enough. This proves Equation (5) in [3].

Now let us check Equation (6) in [3]. It reads as follows: for each  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for a.e.  $(\theta, y)$  we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{\substack{0 \leq i < n \\ |f_i(\theta, y)| < \delta}} \log |f_i(\theta, y)|^{-1} &\leq \varepsilon; \\ \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{\substack{0 \leq i < n \\ \text{dist}(h^i(\theta), S_\theta) < \delta}} \log |\text{dist}(h^i(\theta), S_\theta)|^{-1} &\leq \varepsilon. \end{aligned}$$

The first inequality is simply (38). To obtain the second one, note that  $h$  admits an ergodic a.c.i.p.  $\mu$ , which is equivalent to the Lebesgue measure on  $I_a$ . Then by Birkhoff's Ergodic Theorem, for Lebesgue a.e.  $\theta \in I_a$ , the left hand side of the second inequality is equal to  $\int_{\text{dist}(\theta, S_\theta) < \delta} \log |\text{dist}(\theta, S_\theta)|^{-1} d\mu(\theta)$ . Besides, according to (9), for every measurable set  $E \subset I_a$  and every  $n \geq 0$ ,  $|h^{-n}(E)| \leq C_d |I_a| |E|$ . As a result,  $\frac{d\mu}{d\text{Leb}} \leq C_d |I_a|$  on  $I_a$ . Since  $S_\theta$  is a finite set, it follows that  $\int_{I_a} \log |\text{dist}(\theta, S_\theta)|^{-1} d\mu(\theta) < \infty$ , and hence the second inequality holds when  $\delta > 0$  is small enough.

We have checked that all the conditions of [3, Theorem C] are satisfied provided that  $\alpha$  is small enough. Thus  $F$  has an absolutely continuous invariant measure.

**5.3. Uniqueness of a.c.i.p.** As shown in Lemma 6.1 of [4],  $\Lambda = F^n(I_a \times \widehat{I}_b)$  when  $n \geq 2$ . By a similar argument as in [4, Proposition 6.2], it is easy to prove that  $F$  is topologically exact on  $\Lambda$ . Moreover, by [3, Lemma 5.6], up to a set of zero Lebesgue measure, the basin of each a.c.i.p. of  $F^n$  contains some disk. Therefore the a.c.i.p. of  $F$  is unique.

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